Research statement for Daniel Pomerleano

1. Background and motivating questions

My research centers around algebraic and geometric structures arising in **mirror symmetry**. Mirror symmetry originates in string theory and predicts astonishing and intricate relationships between seemingly disparate notions such as coherent sheaves, which originate in algebraic geometry and Lagrangian submanifolds, which are native to symplectic geometry. More precisely, let X be a smooth projective (or affine) variety. The variety X inherits a natural symplectic structure by restriction of the usual symplectic form on $\mathbb{P}^n$ (or $\mathbb{C}^n$). Kontsevich's Homological Mirror Symmetry conjecture applied to X proposes that there exists a mirror to X, $(\check{X}, w)$, with $\check{X}$ a smooth variety and $w : \check{X} \to \mathbb{A}^1$ a regular function, such that **symplectic invariants** on X are exchanged with **algebraic invariants** on $(\check{X}, w)$. The richest version of this conjecture asserts that we have an equivalence of categories:

$$Fuk(X) \cong MF(\check{X}, w)$$

Here $Fuk(X)$ denotes a suitable *Fukaya category* and $MF(\check{X}, w)$ denotes a category of *matrix factorizations* of w. This conjecture serves as a powerful guiding light for taking structures which are transparent on one side of the mirror and revealing new, unexpected structures on the mirror side. This has lead to the development of tools in both fields that would otherwise have been difficult to imagine (for notable examples see [Bri, GroHacKeeKon, SeiTho]).

My two main research projects to date aim to exploit mirror symmetry to develop new structures in both algebraic and symplectic geometry. The first concerns the **symplectic cohomology ring**, $SH^*(X)$, of affine varieties X. Symplectic cohomology is defined for a wide class of open symplectic manifolds but in general is a very poorly behaved and unwieldy invariant. On the other hand, *mirror symmetry suggests that for affine varieties, $SH^*(X)$ should satisfy very strong algebraic finiteness conditions.* In all known examples [Aur, GroHacKee], the mirror to an affine variety X is a pair $(\check{X}, w)$ such that the naturally defined morphism:

$$\check{X} \to Spec(\Gamma(\mathcal{O}_{\check{X}}))$$

is a proper map and $\Gamma(\mathcal{O}_{\check{X}})$ is a finitely-generated ring. Our first project seeks to explore consequences of this picture of mirror symmetry for the symplectic cohomology ring. There is a natural morphism:

$$\mathcal{CO} : SH^*(X) \cong HH^*(Fuk(X))$$

which is widely expected to be an isomorphism.[1] Mirror symmetry along with the derived invariance of Hochschild cohomology would then imply that:

$$SH^*(X) \cong R\Gamma(\Lambda^\bullet T_{\check{X}}, [w, -])$$

The right hand side of the equation, which may be viewed as the space of functions on the critical locus in a suitable "derived" sense, is a finitely generated ring and so we expect the left hand side to be as well. Moreover, in the case where X is a log Calabi-Yau variety, similar mirror symmetry considerations imply that $SH^0(X)$ should come equipped with a *canonical basis* over the ground ring [GroHacKee]. In examples of interest, these canonical bases are expected to be related to other canonical bases which arise in representation theory and algebraic geometry [GonShe]. The goal of the first project is to answer the following question:

Question 1.1. *Is $SH^*(X)$ finitely generated as a ring? In the case where X is a log Calabi-Yau variety, does $SH^0(X)$ have a canonical basis?*

[1] here we must consider the wrapped Fukaya category of Abouzaid and Seidel

More precisely, we aim to develop new models for computing symplectic cohomology which are specifically designed to answer this and related questions. In fact, we expect that the symplectic cohomology ring can be computed completely algebraically using *punctured invariants*, variants of log Gromov-Witten invariants which are currently being developed by Abramovich, Chen, Gross, Siebert [GroSie]. We describe this program as well as the results we have obtained so far in Section 2.2.1.

My second research project concerns the geometry of the mirror pair $(\check{X}, w)$. For a general projective variety X [2], the critical locus of the function w will typically be quite complicated and in particular non-isolated (see [KapKatzOrlYot] where the case when X is a genus two curve is described in detail). Mirror symmetry states that the mirror to $H^*(X)$ is a vector space $R\Gamma(\Lambda^\bullet\Omega_{\check{X}}, dw\wedge)$, which in these examples may be identified with the above space of functions on the derived critical locus. We may associate to X its Gromov Witten invariants, which may be organized into a *cohomological field theory*:

$$H_*(\overline{\mathcal{M}}_{g,n+m}) \otimes H^*(X)^{\otimes m} \to H^*(X)^{\otimes n}$$

Question 1.2. *Can we intrinsically construct a cohomological field theory with space of states* $R\Gamma(\Lambda^i\Omega_{\check{X}}, dw\wedge)$ *which is mirror to the Gromov-Witten invariants on* X*?*

Besides its intrinsic importance to any formulation of closed string mirror symmetry, an answer to this question would have interesting applications to singularity theory. When $(\check{X}, w)$ has isolated singularities, much work has already been done on Question 1.2, leading in particular to **noncommutative Hodge theory** [KatKonPan], a rich collection of invariants which generalize classical invariants of singularities. Thus, the question can be viewed as attempting to generalize these Hodge theoretic invariants to more complicated singularities. We will describe our progress on this question in Section 2.1.1. We will also see in Section 2.1.2 that it is natural to consider related questions in the equivariant setting where a reductive group G acts on $\check{X}$ as well.

2. Past and future research

2.1. Noncommutative Hodge theory.

2.1.1. *Global Matrix Factorizations.* In joint work with Kevin Lin [**LinPom**], we developed the theory of "hybrid models," categories of matrix factorizations whose critical loci are not necessarily isolated. One major motivation for the development of this theory comes from Katzarkov's programme [KapKatzOrlYot] to extend the Homological Mirror Symmetry conjecture to varieties of general type, where the mirror to such varieties is often a hybrid model. Let Y be a quasi-projective variety and w a non-constant function $w : Y \to \mathbb{A}^1$. We begin by establishing some homological algebra needed to define an appropriate category of matrix factorizations $MF(Y, w)$. Similar definitions were introduced simultaneously by Orlov [Orl]. We then proved the following theorem:

Theorem 2.1. *If* Y *is as above and the critical locus is compact, then* $MF(Y, w)$ *is smooth and proper. If in addition* Y *is Calabi Yau, then the category* $MF(Y, w)$ *is a Calabi Yau dg-category.*

We also prove:

Theorem 2.2. $HH_*(MF(Y, w)) \cong \mathbb{H}^*(\Lambda^i\Omega_Y, dw\wedge)$ *and* $HH^*(MF(Y, w)) \cong \mathbb{H}^*(\Lambda^i T_Y, [w, -])$

Here $\Lambda^i\Omega_Y$ denotes the exterior powers of the sheaf of differential forms and $\Lambda^i T_Y$ denotes exterior powers of the tangent sheaf and $[w, -]$ denotes Schouten bracket with w. The above theorems were proven independently and in a different way by Anatoly Preygel [Pre]. As a consequence of this

[2] particularly in examples where X is of general type

theorem, we obtain a fully extended **topological quantum field theory** (TQFT) associated to the category $\mathcal{C} = MF(Y, w)$. One way of thinking about a TQFT is as an assignment of operations:

$$C_*(\mathcal{M}_{g,m+n}) \otimes Z(S^1)^{\otimes m} \to Z(S^1)^{\otimes n}$$

where $\mathcal{M}_{g,m+n}$ denotes the moduli space of Riemann surfaces with m incoming circles and n outgoing circles in a way which is compatible with glueing operations. We have seen that a proper formulation of mirror symmetry requires the existence of a cohomological field theory, which requires extending these maps to the Deligne-Mumford compactification. A folk-theorem, due independently to Kontsevich and Lurie, gives an elegant description of the extra data that is required to extend a TQFT to the Deligne-Mumford compactification. In particular, the annulus defines a perfect pairing:

$$Z(S^1) \times Z(S^1) \to \mathbb{C}$$

which is invariant under the action of orientation-preserving diffeomorphisms of the annulus. Extending this to a theory defined on the Deligne-Mumford compactifications is then equivalent to promoting this to a pairing which invariant under a larger group, given by the wreath product of $\mathbb{Z}/2\mathbb{Z}$ and $SO(2)$. Equivalently, we can think of choosing a splitting for the *Hodge-de Rham spectral sequence*,

$$HH_*(\mathcal{C})[[u]] \to HC^-_\bullet \tag{1}$$

where $\mathcal{C}$ is the dg-category of boundary conditions and $HC^-_\bullet$ denotes the negative cyclic homology, in a way which is compatible with the above pairing. By [Sab, Pre, **LinPom**], we now know that the above spectral sequence degenerates in the case that $\mathcal{C} = MF(Y, w)$. We must now determine how to choose a splitting which is compatible with the above pairing.

In the case of a Landau-Ginzburg model of the form $(\mathbb{C}[[x_1, x_2, \cdots, x_n]], f)$, where f has isolated singularities, the above pairing is the residue pairing:

$$\frac{\mathbb{C}[[x_1, x_2, \cdots, x_n]]}{(\partial_i f)} \times \frac{\mathbb{C}[[x_1, x_2, \cdots, x_n]]}{(\partial_i f)} \to \mathbb{C}$$

$$(g_1, g_2) \to \int_{|x_i|=\epsilon} \frac{(g_1 \cdot g_2)\Omega_{std}}{\prod_i \partial_i f}$$

The choice of splitting is supplied by Saito's notion of a *good section*, which is a reformulation of his concept of a *primitive form* [Sai]. Thus, the fundamental question is:

Question 2.3. *Can we generalize Saito's construction of good sections to hybrid models ?*

In unpublished work, we have combined the methods of [Ram] with those of [**LinPom**] in order to give an explicit formula for this pairing for arbitrary hybrid models, $MF(Y, w)$. The resulting question can be reduced to a concrete problem in mixed Hodge theory that we are currently investigating.

2.1.2. *Noncommutative Hodge theory of quotient stacks.* In this section we consider the situation where G acts on our variety Y and our potential $w : Y \to \mathbb{A}^1$ is equivariant for this action. We will view this as giving a map from the quotient stack $w : Y/G \to \mathbb{A}^1$. The motivating question is:

Question 2.4. *What is the appropriate notion of noncommutative Hodge structure in these equivariant contexts?*

As we have previously discussed, the spectral sequence (1) plays a central role in noncommutative Hodge theory (it is the analogue of the classical Hodge-to-deRham spectral sequence). We say that a category has the degeneration property if this spectral sequence degenerates. As a first step to answering the question , in joint work with Daniel Halpern-Leistner [**HLPom**], we show that:

Theorem 2.5. [**HLPom**] *Let $w : Y/G \to \mathbb{A}^1$ be an LG-model, where Y is a smooth projective over affine G-scheme such that $\Gamma(Y, \mathcal{O}_{\mathrm{Crit}(w)})^G$* [3] *is finite dimensional, then the degeneration property holds for* $\mathrm{MF}(Y/G, w)$.

We also consider in detail the case when the function $w = 0$, where our underlying category is $\mathbb{Z}$-graded and the notion of noncommutative Hodge structure essentially reduces to the notion of a pure Hodge structure. Let $M \subset G$ be a maximal compact subgroup. In this case, we show that one can recover the equivariant topological K-theory $K_M(Y^{an})$ from the dg-category $\mathrm{Perf}(Y/G)$. The first ingredient is the recent construction by A. Blanc of a topological K-theory spectrum $K^{top}(\mathcal{A})$ for any dg-category $\mathcal{A}$ over $\mathbb{C}$ [Blanc]. Blanc constructs a Chern character natural transformation $\mathrm{ch} : K^{top}(\mathcal{A}) \to HP(\mathcal{A})$, shows that $\mathrm{ch} \otimes \mathbb{C}$ is an equivalence for Perf of a finite type $\mathbb{C}$-scheme, and conjectures this property for any smooth and proper dg-category $\mathcal{A}$. We show that $\mathrm{ch} \otimes \mathbb{C}$ is an isomorphism for all categories of the form $\mathrm{Perf}(\mathcal{Y})$, where $\mathcal{Y}$ is a smooth DM stack or a smooth quotient stack. In fact, we expect that this "lattice conjecture" should hold for a much larger class of dg-categories, such as the categories $D^b(\mathcal{Y})$ for any finite type $\mathbb{C}$-stack and $\mathrm{Perf}(Y/G)$ for any quotient stack. Following some ideas of Thomason in [Tho], we next construct a natural "topologization" map $\rho_{G,X} : K^{top}(\mathrm{Perf}(Y/G)) \to K_M(Y^{an})$ for any smooth G-quasiprojective scheme Y and show:

Theorem 2.6. *([**HLPom**]) For any smooth quasi-projective G-scheme Y, the topologization map and the Chern character provide equivalences* [4]

$$K_M^*(Y^{an}) \otimes \mathbb{C} \xleftarrow{\rho_{G,Y}} \pi_* K^{top}(\mathrm{Perf}(Y/G)) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}} H_* C_\bullet^{per}(\mathrm{Perf}(Y/G))$$

These results enable us to construct a *pure Hodge structure on $K_M(Y^{an})$*. The Hodge structure on $K_M(Y^{an})$ that we obtain in this way has no commutative construction to our knowledge, though it should be noted that C. Teleman [Tel], has shown that a version of the Hodge-de Rham spectral sequence for $H_G^*(Y^{an})$ degenerates for such G-schemes and that the (a priori mixed) Hodge structure on $H_G^*(Y^{an})$ is pure in this case.

2.1.3. *Noncommutative examples, curved string topology, and Symplectic topology.* This project, carried out in [**Pom2**], can be viewed as a bridge between this section and the next section. It has two components, an algebraic component and a symplectic component described below. For affine varieties, $Y = \mathrm{Spec}(A)$, an alternative point of view on matrix factorization is that one is "curving" the algebra A by an element w. This description has the benefit of allowing for noncommutative generalizations, because it also makes sense to curve noncommutative algebras A by a suitably central element w. Let $\mathcal{M}$ denote a compact simply connected manifold whose rational homotopy type is "pure Sullivan."[5] We study the non-commutative dga $C_*(\Omega\mathcal{M})$, where Ω denotes the based-loop space at some point and let w be a suitably central element in this ring. We again define an appropriate category of modules over the curved dga, which we call $MF(C_*(\Omega\mathcal{M}), w)$.

Theorem 2.7 ([**Pom2**]). *There is an explicit criterion for determining when the category $MF(C_*(\Omega\mathcal{M}), w)$ defines a smooth and proper weakly* [6] *Calabi-Yau category.*

Via Koszul duality, this can be viewed as studying certain A_∞-smoothings of "derived singularities" of the form $C^*(\mathcal{M}, \mathbb{C})$.

[3]Our theorem in fact holds for certain smooth quasi-projective varieties G-schemes which admit a "semi-complete KN stratification"

[4]These homology level equivalences are induced by suitable chain maps

[5]This is a rational homotopy theoretic analogue of a complete intersection

[6]In many cases this CY structure can be canonically lifted to HC^-

Next we turn to giving a symplectic intrepretation for some of the algebraic deformations introduced in Theorem 2.7. Let M be a projective variety such that $X = M \setminus D$, where D is smooth ample divisor. It has been known since [Sei5] that one can associate to this data an infinitesimal deformation of $Fuk(X)$ or in other words, a class $\alpha \in HH^*(Fuk(X))$ by counting holomorphic disks that intersect the divisor exactly once. Building upon this idea, we managed to give a geometric interpretation of the above deformation in the case $\mathcal{M}$ is any simply connected Zoll manifold, such as S^n $(n > 1)$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$. Using the technique of symplectic cutting, one can define a canonical compactification M of the open disk bundle $D^*(\mathcal{M})$ by a smooth divisor D. We define a version of Floer cohomology for the zero section L inside of M, which counts disk that intersect the divisor D with multiplicity d. A related concept was developed independently by Sheridan[She] in his proof of the Homological Mirror Symmetry conjecture for Calabi-Yau hypersurfaces in $\mathbb{C}P^n$. In [**Pom2**], we prove the following result:

Theorem 2.8 ([**Pom2**])**.** *The category of perfect modules over the A_∞ algebra $HF^*_d(L, L)$ is equivalent to $MF(C_*(\Omega\mathcal{M}), u^d)$, where u is a canonically defined central element of $H_*(\Omega\mathcal{M})$.*

Assume now that we have a strict normal crossings compactification by ample divisors D_i, e.g. $X = M \setminus \cup_{i=1}^{i=j} D_i$, where the D_i all intersect cleanly. Assume further that M and each $\cap_I D_i$ for each I a subset of $[1, 2, \cdots, j]$ is a monotone symplectic manifold. Based upon heuristic considerations concerning Seidel's construction together with considerations an auxilliary moduli space introduced in the proof of the above theorem, [**Pom2**] defines an algebra $\mathcal{A}$ explicitly in terms of the Gromov-Witten invariants of $\cap_I D_i$, which we expect under good circumstances [7] to be isomorphic to $HH^0(Fuk(X))$. We check in a number of examples that this prediction is consistent with standard conjectures in mirror symmetry.

2.2. **Symplectic topology.**

2.2.1. *Symplectic Cohomology of Affine Varieties.* The computations of Floer theoretic deformations from [**Pom2**] are done by slightly ad-hoc methods, and the present project was initially motivated by the goal of treating these examples more systematically. The key invariant for doing this is the *symplectic cohomology.* This is defined as the cohomology of a complex

$$(\bigoplus_{\sigma_H} \mathbb{C} \cdot |\sigma_H|, d)$$

where H is a time dependent-Hamiltonian with controlled asymptotics at ∞ and σ_H denotes the set of time-one orbits of the Hamiltonian vector field X_H. The differential involves counting solutions to a perturbed J-holomorphic curve equation with prescribed asymptotics. Given the perturbed nature of the equation, it is very hard to compute directly— the only complete computations for affine varieties are for simple cases like $\mathbb{C}^*$, $\mathbb{C}$ and products. By Hironaka's resolution of singularities theorem, for X an affine algebraic variety, there is a projective algebraic manifold M, and an ample normal crossings divisor $\mathbf{D} = \cup D_i$, $i \in \{1, \cdots, k\}$, such that $M - \mathbf{D} \cong X$. Consider the cochain complex:

$$C^*_{log}(M, \mathbf{D}) := \bigoplus_{I \subset \{1,\ldots,k\}} t^{\vec{\mathbf{v}}_I} C^*(\mathring{S}_I, \mathbf{k})[t_i \mid i \in I] \tag{2}$$

Here $\vec{\mathbf{v}}_I \in (\mathbb{N})^k$ is the vector $(v_1, \ldots, v_k)$,

$$v_i := \begin{cases} 1 & i \in I \\ 0 & \text{otherwise.} \end{cases}$$

[7] this may hold for any log CY pair for which all components are ample and all strata monotone; however for more complicated NC compactifications we need a more sophisticated construction as considered below

and $\mathring{S}_I$ is the open torus bundle to the stratum D_I. In particular $\mathring{S}_\emptyset = X$, and $\mathring{S}_I = \emptyset$ if the intersection $\cap_{i\in I} D_i$ is empty. We state the main "meta conjecture":

"Meta Conjecture." [8] There is deformation ∂_{def} of the singular cohomology differential on $C^*_{log}(M, \mathbf{D})$ defined in terms of relative GW invariants of the pair $(M, \mathbf{D})$ such that the complex $(C^*_{log}(M, \mathbf{D}), \partial_{def})$ computes symplectic cohomology.

Our approach to obtaining a precise statement and proof of this Meta Conjecture requires two steps. The first is a geometric one which is largely independent of counting pseudo-holomorphic curves, and the second step involves "virtual counting techniques" which enables us to construct the above deformation. In the first step, which is joint work with Sheel Ganatra, we consider the case when moduli spaces of relative holomorphic curves are empty for geometric reasons. To describe it set

$$H^*_{log}(M, \mathbf{D}) := H^*(C^*_{log}(M, \mathbf{D})) \tag{3}$$

Namely, we define a linear subspace of *admissible classes* $H^*_{log}(M, \mathbf{D})^{ad} \subset H^*_{log}(M, \mathbf{D})$ and show that [**GanPom**]

Theorem 2.9. [**GanPom**] *There is a canonical map* [9]*:*

$$\mathrm{PSS}^+_{log} : H^*_{log}(M, \mathbf{D})^{ad} \to SH^*_+(X) \tag{4}$$

This map is defined by counting pseudoholomorphic thimbles with tangency conditions along $\mathbf{D}$*.*

Our construction is inspired by a map introduced by Piunikhin, Salamon, and Schwarz [PiuSalSch] relating the (quantum) cohomology of M and the Hamiltonian Floer cohomology of a non-degenerate Hamiltonian. For algebraic geometers, it may be helpful to view this construction as giving, currently in special cases where transversality can be achieved, symplectic analogues of the theta functions of [GroHacKee].

There is a wide class of *topological pairs* $(M, \mathbf{D})$ for which the map (4) is particularly well behaved. The key feature of such pairs is that the relevant moduli spaces of relative pseudo-holomorphic curves are all generically empty. A typical example of a topological pair is the case where M is any projective variety and $\mathbf{D}$ is the union of at least $n+1$ generic hyperplane sections, where $n = \dim_{\mathbb{C}}(M)$. In the topological case, we have $H^*_{log}(M, \mathbf{D})^{ad} = H^*_{log}(M, \mathbf{D})$ and moreover there is a canonical lifting of (4) to a *Log PSS morphism*

$$\mathrm{PSS}_{log} : H^*_{log}(M, \mathbf{D}) \to SH^*(X) \tag{5}$$

When the pair $(M, \mathbf{D})$ is not topological, we formulate the obstruction to lifting a given admissible class to $SH^*(X)$ in terms of a Gromov Witten invariant. When this obstruction vanishes, this provides a way to produce *distinguished classes* in $SH^*(X)$ which we will show may be applied to study the symplectic topology of X. Our obstruction theory is inspired by the above meta conjecture and can be viewed as an elementary special case of this conjectural theory.

The existence of this map has some interesting consequences for symplectic topology. Let $h(\mathbf{w})$ be a generic Laurent polynomial in $n-1$-variables and $Z^o \hookrightarrow (\mathbb{C}^*)^{n-1}$ denote the zero locus of $h(\mathbf{w})$. Set X to be the *conic bundle* given by

$$X = \{(u, v, \mathbf{w}) \in \mathbb{C}^2 \times (\mathbb{C}^*)^{n-1} | uv = h(\mathbf{w})\} \tag{6}$$

For simplicity, we assume that Z^0 is connected. The symplectic topology of these varieties (and the closely related conic bundles over $\mathbb{C}^{n-1}$) is very rich and can be approached from different

[8] This statement is not a precise conjecture for many reasons, for example it is likely better to use a Cech model for singular cohomology

[9] SH^*_+ is the cohomology of a certain canonically defined quotient of the complex defining SH^*

perspectives, see for instance [AbouAurKat, Sei2]. For example, there is a standard construction of Lagrangian spheres in X given by taking a suitable Lagrangian disc $j : D^{n-1} \to (\mathbb{C}^*)^{n-1}$ with boundary on the discriminant locus and "suspending" it to a Lagrangian $S^n \hookrightarrow X$ [GroMat, Sei1].

It is natural to ask: *what are the possible topologies of exact Lagrangians in these conic bundles*? The suspension construction typically provides a rich collection of Lagrangian spheres and Seidel [Sei2] has provided constructions of exact Lagrangian tori in certain examples. Our first application is a relatively complete classification of the diffeomorphism types of exact Lagrangian submanifolds in three dimensional conic bundles over $(\mathbb{C}^*)^2$.

Theorem 2.10. [**GanPom**] *Let X be a three-dimensional conic bundle over $(\mathbb{C}^*)^2$ of the form* (6)*, and let $Q \hookrightarrow X$ be a closed, oriented, exact Lagrangian submanifold of X. Then Q is either diffeomorphic to T^3 or $\#_n S^1 \times S^2$ (by convention, the case $n = 0$ corresponds to S^3).*

By combining our methods with those from [Sei3], we also prove the following result concerning disjoinability of Lagrangian spheres:

Theorem 2.11. [**GanPom**] *Let $\mathbf{k}$ be a field and $n \geq 3$ be an odd integer. Suppose that X is a conic bundle of the form* (6) *of total dimension n over $(\mathbb{C}^*)^{n-1}$ and that $Q_1, \cdots, Q_r$ is a collection of embedded Lagrangian spheres which are pairwise disjoinable. Then the classes $[Q_1], \cdots, [Q_r]$ span a subspace of $H_n(X, \mathbf{k})$ which has rank at least $r/2$.*

We now turn to the natural question of when the map (5) is an isomorphism. In work in preparation with Ganatra we expect to prove that:

Theorem 2.12. [**GanPom3**] *There is a multiplicative* [10] *spectral sequence*

$$H^*_{log}(M, \mathbf{D}) => SH^*(X) \tag{7}$$

For topological pairs, this spectral sequence degenerates and the map (5) *is an isomorphism of rings.*

This provides a wide class of new calculations of symplectic cohomology rings. A proof of Theorem 2.12 in the simplest case when $\mathbf{D} = D$ is a smooth divisor appears in [**GanPom2**]. This already produces a large class of new examples of symplectic cohomology rings. The normal crossings case is much more involved, but the important ingredient here is McLean's paper [McL] to develop a convenient model for symplectic cohomology. In a neighborhood of the compactifying divisors, McLean constructs a contact manifold whose Reeb flow orbits maybe identified with the above $\mathring{S}_I$. [11]

An elaboration of McLean's techniques enables one to produce a Hamiltonian adapted to the compactification which forthcoming work of Borman and Sheridan [BorShe] *show may be used to compute $SH^*(X)$.* [12] Their adapted Hamiltonians give us enough control to produce a local solution to the PSS equation which enables us to prove the map is an isomorphism. Using the above theorem, we may also easily give an answer to the first part of Question 1.1 for a wide class of pairs :

Corollary 2.13. [**GanPom3**] *For any pair $(M, \mathbf{D})$ which is log CY or log general type, the $SH^*(X)$ is a finitely generated ring.*

Now we turn to the "virtual part" of the story which will enable us to plug back in the pseudo-holomorphic curves which are absent for topological pairs. Our main tool here shall be recent work of Pardon [Par1], who develops a beautiful and flexible framework of implicit atlases for equipping

[10] the multiplicative structure on $H^*_{log}(M, \mathbf{D})$ is defined by a simple topological formula

[11] McLean must first deform the divisors to be symplectically orthogonal in order to achieve this

[12] This is one of the steps which is immediate in the case $\mathbf{D} = D$, but which requires more work in the normal crossings case.

moduli spaces of pseudoholomorphic curves with virtual cycles. Fix two multiplicities $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 \in \mathbb{N}^k$. By performing a *heuristic* bubbling off analysis for (virtual) dimension one components of all of the moduli-space involved in defining the map PSS_{log}, it is not difficult to construct a moduli space $\mathcal{M}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ of relative pseudoholomorphic curves which should govern the quantum corrections to the differential on $C^*_{log}(M, \mathbf{D})$. To make this precise, we must answer:

Question 2.14. *Can we construct a compactification* $\overline{\mathcal{M}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)}$ *of* $\mathcal{M}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ *such that* $\overline{\mathcal{M}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)}$ *has the structure of an implicit atlas with boundary?*

This is again easiest in the case when $\mathbf{D} = D$ is smooth, when the compactification can be constructed using SFT compactness. The construction of the implicit atlas structure in this case reduces to a delicate gluing theorem, however there are similar gluing theorems in the literature [Par2]. I have been discussing this gluing problem with Dan Cristofaro-Gardiner. In the normal crossings case, there is much less literature to draw on, although based upon [Ion], we have a conjectural picture for what one possible compactification may look like.

With Mark Gross, we have been working on writing down a version of the above complex, $SH^*_{alg}(M, \mathbf{D})$, which essentially follows the above steps, but whose differential is defined completely algebraically. The idea of this project is to replace the above relative moduli counts with Abramovich and Chen, Gross and Siebert's theory of *punctured invariants*. Ongoing work of Gross and Siebert constructs an algebraic version of $SH^0_{alg}(M, \mathbf{D})$ for all log-Calabi-Yau pairs using these invariants and it seems likely that this construction can be modified to account for higher degree symplectic cohomology classes as well (see page 5 of [GroSie] for related discussion). The main question here is:

Question 2.15. *For any log Calabi-Yau pair* $(M, \mathbf{D})$ *can we construct a vector space* $SH^*_{alg}(M, \mathbf{D})$*, which is a finite rank module over* $SH^0_{alg}(M, \mathbf{D})$ *and such that there is a spectral sequence*

$$H^*_{log}(M, \mathbf{D}) => SH^*_{alg}(M, \mathbf{D}) \tag{8}$$

Once this complex is written down, it is of course interesting to try to compare the algebraic and symplectic versions of these theories. There are of course many fascinating examples that one could explore, for example affine deformations of (multiplicative) quiver varieties, the examples of [GonShe], or those considered in [GroHacKeeKon]. There also should be open string versions of the above theory. Here it is reasonable to speculate that when X is log Calabi-Yau, the wrapped Fukaya category $WFuk(X)$ is a Calabi-Yau category over $SH^0(X)$ in the sense of [BezKal].

2.2.2. *SYZ mirror symmetry for local Calabi-Yau varieties.* The **Strominger-Yau-Zaslow(SYZ)** programme is a comprehensive approach to understanding both the construction of mirror pairs and Kontsevich's HMS conjecture systematically. In joint work with Kwokwai Chan and Kazushi Ueda [**ChaPomUed**], we began the implementation of the SYZ programme in the case when X is a smoothing of the conifold singularity, the hypersurface defined by the equation $xy - zw = 1$. This example has been a very important one throughout the development of the subject.

Starting from a special Lagrangian fibration on $X \to \mathbb{R}^3$, we use wall-crossing techniques to construct the mirror $\check{X}$ as a certain open subvariety of the total space of the vector bundle $Y := Tot(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \to \mathbb{C}P^1$. We then construct Lagrangian sections L_i of the SYZ fibration and certain Lagrangian S^3, S_i which fiber over a affine interval of the SYZ base. We prove the following theorem:

Theorem 2.16. [**ChaPomUed**] *The SYZ transform of* L_i *is* $\mathcal{O}(i)$ *and the SYZ transform of* S_i *is* $\mathcal{O}_{\mathbb{C}P^1}(-i)$*.*

We then compute wrapped Floer homology for these Lagrangians. More precisely, we prove the following theorem:

Theorem 2.17. [**ChaPomUed**] *The SYZ transform gives rise to fully-faithful embeddings*

$$D^bCoh(\check{X}) \to WFuk(X) \quad D^bCoh_{\mathbb{C}P^1}(\check{X}) \to Fuk(X)$$

We expect that these Lagrangian sections generate the wrapped Fukaya category, which would provide a complete proof of homological mirror symmetry in this example.

It is interesting to consider the conic bundles X from (6) from this perspective as well. Namely, recent work of Abouzaid-Auroux-Katzarkov has enabled us to revisit these examples from the point of view of the Strominger-Yau-Zaslow conjecture [13]. There is a canonical S^1 action on X. Let μ denote the moment map. We can write down symplectomorphisms:

$$\phi_\mu : X_\mu/S^1 \to (\mathbb{C}^*)^n$$

$$(log|\phi_\mu|, \mu) : X \to \mathbb{R}^n \times \mathbb{R}$$

The SYZ mirror can be constructed using wall-crossing techniques and determined to be an open subset of a toric variety Y whose affinization is described by the cone over the Newton polytope of $f(z_1, \cdots, z_n)$. The toric variety Y is given by a polarized unimodular triangulation of the Newton polytope, which is equivalent to a choice of tropical degeneration for the hypersurface defined by $f(z_1, \cdots, z_n) = 0$. Finally, Y is equipped with a canonical regular function $h : Y \to \mathbb{A}^1$ which vanishes to first order along each of the toric divisors. The main result of Abouzaid, Auroux, Katzarkov [AbouAurKat] is the following theorem:

Theorem 2.18. *$\check{X}$ is isomorphic to $Y \setminus h^{-1}(1)$.*

Remark 2.19. In unpublished work, we have also used the techniques of Abouzaid, Auroux, Katzarkov to construct mirrors to *hypertoric varieties,* e.g. hyperkahler quotients of the form $\mathbb{C}^n///\mathbb{T}^d$. Namely, on any hypertoric variety X there is a complementary $\mathbb{T}^{n-d}$ which acts symplectically on X. We have an indentification

$$\mu_{\mathbb{C}} : X//\mathbb{T}^{n-d} \cong \mathbb{C}^{n-d}$$

given by the algebraic moment map. Using the approach of Abouzaid, Auroux, Katzarkov, we can construct an SYZ fibration and construct a mirror $\check{X}$. All of the questions raised in this section have analogues for these examples as well.

In joint work 'with Chan and Ueda [**ChaPomUed2**] , we have used the methods of [Abou2] to construct Lagrangian sections L_σ, whose Hamiltonian isotopy classes are in bijection with line bundles on the toric variety Y as well as Lagrangian spheres S_σ which are fibered over bounded components of the complement of the tropical amoeba $\mathbb{R}^n - \Pi$ and are in bijection with the push-forwards of line bundles from the corresponding compact divisor. We describe a version of wrapped Floer theory, $WF^*_{ad}(L, L)$, where L is a certain class of admissible Lagrangians, using certain admissible Hamiltonians H_{ad}. These Hamiltonians are modelled on pullbacks of a proper convex function on the base:

$$H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

Our main theorem is the following theorem:

Theorem 2.20. [**ChaPomUed2**] *Let L_0 be the zero section of the fibration, There is an isomorphism*

$$WF^*_{ad}(L_0, L_0) \cong \Gamma(\mathcal{O}_{\check{X}}) \tag{9}$$

[13] In the SYZ mirror construction, the variety X is equipped with a modified S^1 invariant symplectic form, which is symplectomorphic to the standard symplectic form restricted from affine space. We shall surpress this technical point from our discussion.

Furthermore there is an isomorphism $WF^*_{ad}(L_0, L_0) \cong WF^*(L_0, L_0)$, where $WF^*(L_0, L_0)$ is the standard wrapped Floer cohomology.

The wrapped Floer cohomology ring on the left hand side of (9) comes with a natural basis given by Hamiltonian chords. As suggested by Tyurin and emphasized by Gross, Hacking, Keel, the images of these Hamiltonian chords on the right hand side of (9) give generalizations of theta functions on abelian varieties. We use this result to calculate the zero-th symplectic cohomology $SH^0(X)$ of X. Abouzaid [Abou1] has introduced a map

$$\mathcal{CO}\colon SH^0(X) \to WF^0(L_0). \tag{10}$$

We prove the following theorem:

Theorem 2.21. [**ChaPomUed2**] *The map* (10) *is an isomorphism.*

Finally, we apply all of these calculations to give a proof of homological mirror symmetry in the simplest case when our polynomial $h(w_1, w_2)$ is $1 + w_1 + w_2$ (this corresponds to the case when the mirror $\check{X}$ is $\mathrm{Spec}(\mathbb{C}[x, y, z][(xyz - 1)^{-1}])$).

Theorem 2.22. [**ChaPomUed2**] *L_0 generates the wrapped Fukaya category of Y. In particular, there is an equivalence*

$$\psi\colon D^bWFuk(X) \cong D^bcoh(\check{X}) \tag{11}$$

of enhanced triangulated categories sending L_0 to $\mathcal{O}_{\check{X}}$.

We further generalize these calculations to finite covers, which correspond to taking finite quotients on the mirror side using a type of McKay correspondence. This includes the case when the toric variety $Y = \mathcal{K}_{\mathbb{P}^2}$. An obvious question is:

Question 2.23. *Can we use the SYZ picture to prove Homological Mirror Symmetry for more general examples? In a different direction, can we use the SYZ picture to generalize Seidel and Thomas' work on symplectic mirrors to derived autoequivalences for more general toric Calabi-Yau varieties?*

Another interesting question which arises is:

Question 2.24. *Can we understand the mirror to the derived McKay correspondence for three dimensional crepant resolutions in symplectic terms?*

In two dimensions, the picture is very clear, and I hope to extend this picture to these examples.

More generally, we expect to be able to construct Lagrangian submanifolds which are sections of the SYZ fibration over each tropical cell of the tropical amoeba of the tropical limit of f. It is easy to do this when the tropical hypersurface is a curve in $\mathbb{R}^2$. The SYZ transform of such a Lagrangian is expected to be a line bundle supported on a lower dimensional toric stratum of $\check{X}$. In view of what follows, it will also be important to extend our construct to include certain spherically fibered co-isotropic branes associated to tropical subcomplexes. Recent advances in the theory of variation of GIT quotients give rise to the construction of new derived autoequivalences of $D^bCoh(\check{X})$, the *generalized spherical twists* about coherent sheaves supported on lower dimensional toric strata. The GIT quotient depends on a parameter θ, which is equivalent to the polarized unimodular triangulation of the Newton polytope needed to construct the tropical degeneration of the hypersurface. This suggests that *these generalized spherical twists are mirror to fibered Dehn twists about the Lagrangian (or co-isotropic) cycles constructed via tropical geometry.* A key ingredient in carrying out this work is a generalization for fibered Dehn twists of Seidel's long exact sequence due to Werheim-Woodward [WerWoo]. The simplest case is discussed in the last section of [**ChaPomUed**], where the corresponding generalized spherical twists have been computed explicitly by Segal and Donovan [DonSeg].

2.3. **Quantitative Weinstein conjecture.** This final project is independent of all of the above and is a new direction of research joint with Dan Cristofaro-Gardiner and Michael Hutchings. The three-dimensional case of the Weinstein conjecture asserts that every contact form on a closed three-manifold has at least one Reeb orbit. This was proved by Taubes in 2006. This result naturally leads to the following question:

Question 2.25. *What can one say about the number of simple Reeb orbits of a contact form on a closed three-manifold?*

Without any further assumptions on the contact manifold Y or the contact form λ, a definitive result in this direction was proven by Cristofaro-Gardiner and Hutchings, who have shown that there are at least two simple Reeb orbits. The lower bound of two is the best possible without further assumptions, because there exist contact forms on S^3 with exactly two simple Reeb orbits. One can also take quotients of these examples by cyclic group actions to obtain contact forms on lens spaces with exactly two Reeb orbits. In order to obtain stronger results, a standard assumption is that λ is *nondegenerate* [14], in order to make direct connection with the powerful theory of holomorphic curves in the *symplectization* $Y \times \mathbb{R}$ [15]. An important result in this direction is:

Theorem 2.26. *(Hutchings-Taubes* [HutTau]*) Let Y be a closed three-manifold which is not S^3 or a lens space. Then every nondegenerate contact form on Y has at least three simple Reeb orbits.*

Theorem 2.26 is proven using *embedded contact homology* (ECH) a three manifold invariant which is defined using pseudoholomorphic curve theory. With Cristofaro-Gardiner and Hutchings, we expect to prove the following major generalization of a result of Hofer-Wyzocki-Zehnder [HWZ2, Cor. 1.10]:

Theorem 2.27. [**CGHutPom**] *Let Y be a closed connected three-manifold and let λ be a nondegenerate contact form on Y. Assume that $c_1(\xi)$, where ξ is the contact 2-plane field, is torsion in $H^2(Y;\mathbb{Z})$. Then there are either two or infinitely many simple Reeb orbits.*

The first main idea of the argument is to study the asymptotics of $\mathrm{ECH}(Y, \lambda)$ to produce a one-dimensional family of embedded genus zero curves $C \subset Y$ which foliate $Y \setminus \cup_i \gamma_i$, where γ_i is a collection of Reeb orbits. The second key idea is to combine this with this results of [HWZ1] to show that this gives rise to a *global surface of section*, which enables one to transfer dynamical questions on Y to questions about fixed points of homeomorphisms of surfaces. From here, Theorem 2.27 follows from known results. Motivated by the main result of [CGHut], it is natural to ask:

Question 2.28. *Can we remove the nondegeneracy hypotheses from Theorem 2.27?*

Discussions with Cristofaro-Gardiner suggest that we may be able to. We may, as in [CGHut], take a very small perturbation $\lambda_f = (1 + f)\lambda$ with $|f|_{C^\infty}$ small to make the contact form nondegenerate. A more delicate version of the analysis involved in Theorem 2.27 may still show that one obtains a suitable family of curves on Y by studying the ECH of the perturbed contact form. Work of [HWZ1] seems to then provide the analytic foundations needed to construct a global surface of section from these curve by taking the limit as $|f|_{C^\infty} \to 0$.

References

[Abou1] M. Abouzaid. A geometric criterion for generating the Fukaya category, 2010, Publications Mathematiques de l'IHES 112, no 1, 191-240.

[Abou2] M. Abouzaid. Morse homology, tropical geometry, and homological mirror symmetry for toric varieties, Selecta Math. (N.S.), 2009, vol. 15, 189–270.

[14] This is an analogue of the Morse condition for contact forms

[15] Remarkably the main result of [CGHut] is also proven using pseudoholomorphic curve theory by first perturbing to a nondegenerate form and then employing a subtle limiting argument

[AbouAurKat] M. Abouzaid, D. Auroux, L. Katzarkov, Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces, arXiv:1205.0053.

[Aur] D. Auroux. Mirror symmetry and T-duality in the complement of an anticanonical divisor, 2007, `arxiv:math/0706.3207`

[BezKal] R. Bezrukavnikov, D. Kaledin, McKay equivalence for symplectic resolutions of singularities, 2004, `arxiv:math/0401002`

[Blanc] Anthony Blanc, Topological K-theory of complex noncommutative Spaces, 2012, to appear in Compositio Mathematica.

[BorShe] S. Borman, N. Sheridan, in preparation, 2016.

[Bri] T. Bridgeland, Stability Conditions on Triangulated Categories, Annals of Mathematics, **166** (2007), 317–345.

[**ChaPomUed**] K. Chan, D. Pomerleano, K. Ueda. Lagrangian Torus Fibrations and Homological Mirror Symmetry for the Conifold, Comm. Math. Phys. 341 (2016), no. 1, 135-178.

[**ChaPomUed2**] K. Chan, D. Pomerleano, K. Ueda, Lagrangian sections on mirrors of toric Calabi-Yau 3-folds, 62 pages, arxiv:1602.07075, *preprint available from author's website.*

[CGHut] D. Cristofaro-Gardiner and M. Hutchings, *From one Reeb orbit to two*, J. Diff. Geom. **102** (2016), 25–36.

[**CGHutPom**] D. Cristofaro-Gardiner, M. Hutchings and D. Pomerleano, *Torsion contact structures have two or infinitely many Reeb orbits*, in preparation, 2016.

[Dio] L. Diogo. Stanford University Thesis, 2012.

[DonSeg] W. Donovan and Ed Segal. Braid Group actions from Surface singularities, in preparation.

[**GanPom**] S. Ganatra, D. Pomerleano, A log PSS map with applications to Lagrangian embeddings, 55 pages, 2016 *preprint available from author's website.*

[**GanPom2**] S. Ganatra, D. Pomerleano, Remarks on symplectic cohomology of smooth divisor complements, 19 pages, 2016 *draft available from author's website.*

[**GanPom3**] S. Ganatra, D. Pomerleano, The log PSS map is an isomorphism, in preparation, 2016.

[GonShe] A. Goncharov, L. Shen, Geometry of canonical bases and mirror symmetry, Inventiones, 202(2015), no.2, 487-633.

[GroHacKee] M. Gross, P. Hacking, S. Keel, Mirror symmetry for log Calabi-Yau surfaces I, 2011, `arxiv:math/1106.4977`

[GroHacKeeKon] M. Gross, P. Hacking, S. Keel, and Maxim Kontsevich, preprint arXiv:1411.1394, 136pp.

[GroMat] M.Gross, D. Matessi, On homological mirror symmetry of toric Calabi-Yau three-folds, 2015.

[GroSie] M. Gross, B. Siebert, Intrinsic mirror symmetry and punctured Gromov-Witten invariants, 2016, `arxiv:math/1609.00624v2`

[**HLPom**] D. Halpern-Leistner, D. Pomerleano, Equivariant Hodge theory and noncommutative geometry , 46 pages, arxiv:1507.01924.

[HWZ1] H. Hofer, K. Wysocki and E. Zehnder, *The dynamics on three-dimensional strictly convex energy surfaces*, Ann. Math. **148** (1998), 197–289.

[HWZ2] H. Hofer, K. Wysocki and E. Zehnder, *Finite energy foliations of tight three-spheres and Hamiltonian dynamics*, Ann. Math. **157** (2003), 125–257.

[HutTau] M. Hutchings and C. H. Taubes, The Weinstein conjecture for stable Hamiltonian structures, Geometry and Topology 13 (2009), 901–941.

[Ion] E. Ionel, GW invariants relative normal crossings divisors, 2011, arXiv:1103.3977.

[KonSoi] M. Kontsevich, Y. Soibelman. Wall-crossing structures in Donaldson-Thomas invariants, integrable systems and Mirror Symmetry, 2013, `arxiv/1303.3253`

[KapKatzOrlYot] Kapustin, Katzarkov, Orlov, Yotov. Homological Mirror Symmetry for manifolds of general type, preprint, 2010.

[KatKonPan] L. Katzarkov, M. Kontsevich, T. Pantev. *Hodge theoretic aspects of mirror symmetry.* Proceedings of Symposia in Pure Mathematics vol. 78 (2008), *From Hodge theory to integrability and TQFT: tt^*-geometry*, eds. Ron Y. Donagi and Katrin Wendland, 87–174.

[**LinPom**] K. Lin, D.Pomerleano. Global Matrix Factorizations. Math Research Letters, Vol. 20, 91-106. 2013.

[McL] M. McLean. The growth rate of symplectic homology and affine varieties, 2010`arxiv:math/1011.2542`

[Orl] D. Orlov. Matrix factorizations for nonaffine Landau-Ginzburg models, 2011 `arxiv:1101.4051`

[Par1] An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves. Geom. Topol. 20 (2016), no. 2, 779–1034

[Par2] Contact homology and virtual fundamental cycles. Preprint. 2015.

[PiuSalSch] S. Piunikhin, D. Salamon, M. Schwarz, Symplectic Floer-Donaldson theory and quantum cohomology, Contact and symplectic geometry, (Cambridge, 1994),171–200, vol. 8, 1996.

[**Pom**] D. Pomerleano. Curved String Topology and Tangential Fukaya categories, Proceedings of the Strings Math 2011 Conference.

[**Pom2**] D. Pomerleano. Curved String Topology and Tangential Fukaya categories II, Ph.D. Thesis, available: arxiv:math/1208.4881

[Pre] A. Preygel. Thom–Sebastiani and Duality for Matrix Factorizations, 2011, arxiv:math/1101.5834

[Ram] A. Ramadoss. The Mukai pairing and integral transforms in Hochschild homology. New York J. Math. 14 (2008) 643717.

[Sab] C.Sabbah. On a twisted DeRham complex, II. 2010, arxiv:math/1012.3818

[Sai] K. Saito. Period Mapping Associated to a Primitive Form. Publ. RIMS, Kyoto Univ. 19 (1983), 1231-1264

[Sei1] Paul Seidel, Suspending Lefschetz fibrations, with an application to local mirror symmetry, Comm. Math. Phys., vol. 297, no. 2, 2010, 515-528

[Sei2] Paul Seidel, Lectures on categorical dynamics, Author's website, 2013.

[Sei3] Paul Seidel, Disjoinable Lagrangian spheres and dilations, Invent. Math., vol. 197, no. 2, 299–359, 2014.

[Sei5] P. Seidel. Fukaya categories and Deformations, 2002, arxiv:math/0206155

[SeiTho] P. Seidel and R.Thomas, Braid group actions on derived categories of coherent sheaves, Duke Math J. 108 (2001), no. 1, 37-108.

[She] N. Sheridan. Homological Mirror Symmetry for Calabi-Yau hypersurfaces in projective space. Invent. Math., vol. 199, no. 1, 1-186,

[Tel] Constantin Teleman, The quantization conjecture revisited ,Annals of Mathematics, vol. 152, no. 1, 1–43, 2000.

[Tho] Robert Thomason, Equivariant algebraic vs. topological K-homology Atiyah-Segal-style, Duke Mathematical Journal, vol. 56, no. 3, 589–636, 1988.

[WerWoo] K. Werheim, C. Woodward. Exact triangle for fibered Dehn twists, available at http://math.mit.edu/ katrin/papers/tri.pdf