Remarks on symplectic cohomology of smooth divisor complements

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Abstract. Let M be a smooth projective variety and $\mathbf{D}$ an ample simple normal crossings divisor. Under "topological" assumptions on the pair $(M, \mathbf{D})$, the authors have introduced a map from a vector space which we term the *logarithmic cohomology* of $(M, \mathbf{D})$ to symplectic cohomology of the complement $X = M \backslash \mathbf{D}$ [GP2]. In this short note, we prove that a slight modification of this map is an isomorphism when $\mathbf{D} = D$ is a smooth divisor. This enables us to compute the symplectic cohomology as a ring for any such pair. Using the same techniques, we also compute $SH^0(X)$ as a ring for any pair with $\mathbf{D} = D$ a smooth anticanonical divisor.

1. Introduction

Let M be a smooth projective variety and suppose that $i : D \to M$ is a smooth ample divisor on M such that

$$\mathcal{K}_M \cong \mathcal{O}(-mD) \tag{1.1}$$

for some m. Given such an embedding, we may associate a normal bundle to D inside of M which we denote by $\pi : ND \to D$. Equip M with a Kahler form ω corresponding to a positive Hermitian metric on $\mathcal{O}(D)$. Denote by $X := M \setminus D$ and the inclusion map $j : X \to M$. In such a situation, we can define a symplectic cohomology ring $SH^*(X)$ [FH, V, CFH]. The goal of the present note is to make (partial) computations of the ring structure on $SH^*(X)$ using the techniques of [GP2]. To state our main result, we fix our ground ring to be a field of characteristic zero $\mathbf{k}$. Consider the vector space

$$H^*_{log}(M, D) := H^*(X) \bigoplus_{\mathbf{v} \in \mathbb{Z}^+} H^*(SD)t^{\mathbf{v}} \tag{1.2}$$

The vector space $H^*_{log}(M, D)$ has a cohomological grading given by

$$|\alpha t^{\mathbf{v}}| = |\alpha| + 2(1-m)\mathbf{v} \tag{1.3}$$

and a natural graded-commutative ring structure respecting this grading (see 2.2 for the definition). In [GP2], we introduced the notion of a topological pair (M, D) [1] which enabled

[1] The definition introduced in that paper was for general simple normal crossings divisors and is slightly weaker in the case of a single smooth divisor than that introduced here. However the analysis needed to construct the PSS map carries over without any changes.

us to define canonical classes in symplectic cohomology from classes in $H^*_{log}(M, D)$. This short note is concerned with a map

$$\text{PSS}_{log} : H^*_{log}(M, D) \to SH^*(X) \tag{1.4}$$

which is a slight variant [2] of the map introduced in [**GP2**]. The main theorem of this note is the following:

THEOREM 1.1. *Suppose that (M, D) is a topological pair. The map (1.4) is an isomorphism of rings.*

It is worth emphasizing that the main interest of Theorem 1.1 is in the ring structure. In most examples where Theorem 1.1 applies, the additive structure can be calculated using more traditional spectral sequence methods. Forthcoming work of the authors [**GP**] will prove a generalization of Theorem 1.1 to the simple normal crossings setting. In this note, we limit ourselves to the smooth setting because it is technically simpler and still provides a wealth of new examples.

As also remarked in [**GP2**], even outside of the topological setting, the PSS map may still be used to define canonical elements in symplectic cohomology. For example, when $m = 1$ and $\mathbf{k} = \mathbb{C}$, Kontsevich's homological mirror symmetry conjecture predicts (in somewhat simplified terms) that there is a mirror partner to X, $X^\vee$, which admits a proper algebraic map

$$s_1 : X^\vee \to \mathbb{A}^1_{\mathbf{k}} \tag{1.5}$$

THEOREM 1.2. *There are canonically defined elements $s_{\mathbf{v}}$ together with an isomorphism*

$$\text{PSS}_{log} : \bigoplus_{\mathbf{v} \in \mathbb{N}^{\geq 0}} \mathbf{k} \cdot s_{\mathbf{v}} \cong SH^0(X, \mathbf{k}) \tag{1.6}$$

Moreover as a ring we have that

$$\mathbf{k}[s_1] \cong SH^0(X, \mathbf{k}). \tag{1.7}$$

This theorem could be read as saying that "degree zero piece of (1.4) is still defined and an isomorphism." It is worth noting that in this case, even the additive result does not seem to be achievable by purely spectral sequence theoretic techniques. Regarding the multiplicative structure, we warn the reader that the map (1.7) does not agree with the obvious map sending $s_{\mathbf{v}} \to (s_1)^{\mathbf{v}}$. In fact, this isomorphism contains rich enumerative geometry, worthy of further exploration. More precisely, the elements $s_{\mathbf{v}}$ correspond to canonically defined degree $\mathbf{v}$ polynomials.

It is not difficult to see from our construction that the coefficients of these polynomials are likely defined in terms of certain generalizations of relative Gromov-Witten invariants of the pair (M, D). Proving this rigorously, however, would depend on suitable forms of transversality and gluing that go beyond the scope of the present note. We give some demonstrations of this in elementary examples where such transversality results can be shown to hold.

Though these results are certainly within the range of other approaches, notably the long-term project [**D**, **DL**], we hope that this note illustrates how such calculations can be cleanly performed using the techniques of [**GP2**]. To compute $SH^*(X)$ for arbitrary pairs (M, D) satisfying (1.1) one needs sufficiently strong transversality and gluing results

[2] In this note, we use slightly different families of Hamiltonian functions in our construction. We expect the two constructions to coincide, however we do not address this.

to modify $H^*_{log}(M, D)$ so that (1.4) is well-defined. From there, it seems likely that the arguments from Section 3 should carry over without essential difficulty.

2. Symplectic cohomology and log PSS map

The bundle $\pi : ND \to D$ has a natural Hermitian structure and we let SD denote the associated circle bundle. Let r denote the radial coordinate on ND. The induced linear-connection on ND gives rise to a decomposition into horizontal and vertical pieces, that is to say for any point $p \in ND$, we have a decomposition

$$T_pND = T_pH \oplus T_pV \tag{2.1}$$

where we have an identification

$$T_pV \cong T\mathbb{C} \tag{2.2}$$

Let ω_D denote the restriction of the symplectic form ω to D. Let θ be the trangression one-form on ND which is compatible with the Hermitian structure i.e. such that for any point p we have that

$$d\theta_{|T_pH} = -\pi^*(\omega_D). \tag{2.3}$$

For a fixed $\delta > 0$, let $UD_{2\delta}$ denote the open subset of ND where $r \leq 2\delta$. For δ sufficiently small, the symplectic tubular neighborhood theorem gives an identification $\psi : UD_{2\delta} \to U \subset M$ with $D\psi_{|D} = \mathrm{Id}$ and such that

$$\psi^*(\omega) = \frac{1}{2}dr^2\theta + \pi^*\omega_D$$

where r is the radial coordinate on $UD_{2\delta}$. Observe that we may write $\psi^*(\omega) = d\hat{\theta}$, where $\hat{\theta} = (\frac{1}{2}r^2 - 1)\theta$. We now proceed with the definition of the symplectic cohomology group $SH^*(X)$. For some fixed $r_0 \leq 2\delta$, we will denote the region of U where $r \leq r_0$ by U_{r_0}. Choose $\epsilon << \delta$. Consider a function $h(r^2) : U \to \mathbb{R}^{\geq 0}$ such that $h(r^2) = c - 1/2r^2$ for $0 < c << 1$ on U_ϵ and which vanishes in $U \setminus U_\delta$.

Definition 2.1. *For any slope $\lambda \geq 0$, we say that a Hamiltonian $H_\lambda : M \to \mathbb{R}$ has slope λ if $H_\lambda = \lambda h(r^2)$ on U_ϵ.*

It follows that in U_ϵ, the Hamiltonian vector field

$$X_{H_\lambda} = -\lambda\partial_\theta \tag{2.4}$$

Definition 2.2. *For any open set $U' \subset U$, we say that a compatible almost complex structure J on M is split inside of U' if*

- *it respects the decomposition 2.1 at each point $p \in U'$.*
- *$\pi : U' \to D$ is holomorphic for some ω_D-compatible holomorphic structure on D.*
- *furthermore under the identification 2.2 the complex structure goes to the standard complex structure.*

Definition 2.3. *We say that a compatible almost complex structure $J \in \mathfrak{J}(M)$ is admissible at infinity if it is split inside of U_ϵ. We denote the space of such complex structures by $\mathfrak{J}_c(M, D)$.*

Definition 2.4. *We say that a pair (M, D) is topological if for generic $J \in \mathfrak{J}_c(M, D)$*

- *there are no non-constant J-holomorphic spheres entirely contained in D and*
- *every J-holomorphic sphere intersects D in at least three points.*

Set $\bar{X} = \overline{X \setminus U_\delta}$. For any Hamiltonian function $H : S^1 \times X \to \mathbb{R}$, a fixed point of a time-one flow of the Hamiltonian vector field X_H associated with H is called a *time-one Hamiltonian orbit* of H. The set of time-one Hamiltonian orbits of H will be denoted by $\mathcal{X}(X; H)$. In the specific case under investigation, we work with generic λ so that there are no time one orbits of H_λ inside of U_ϵ. We may choose a small S^1-dependent perturbation $H_\lambda^{\text{pert}} \colon S^1 \times X \to \mathbb{R}$ supported in a neighborhood of the union of all time-one Hamiltonian orbits of H_λ, so that any time-one Hamiltonian orbit of the perturbed Hamiltonian $H_{\lambda,t} := H_\lambda + H_\lambda^{\text{pert}}$ is non-degenerate. We will need to impose slightly more structure on H_λ for our intended applications. Namely, we will assume that

- $H_\lambda = 0$ on $\bar{X}$
- On U, $H_\lambda = h(r^2)$ with $h'(r^2) \leq 0$
- $h''(r^2) \geq 0$.

After appropriately choosing H_λ, we may assume that the orbits of H_λ in X come in isolated families of two types:

- constant orbits $\mathcal{F}_0$ in $\overline{X}$
- orbit sets $\mathcal{F}_d$ which are isomorphic to SD which around the divisor d times for $1 \leq d < \lambda$.

We may assume choose isolating sets U_0 and U_d assume that our functions

$$H_\lambda^{\text{pert}} = \sum_i \tau H_i^{\text{pert}} \tag{2.5}$$

where H_i^{pert} are "Morse-Bott" perturbations supported in U_d and U_0 respectively and τ is sufficiently small. Let $\mathcal{X}_0(X; H_{\lambda,t})$ and $\mathcal{X}_d(X; H_{\lambda,t})$. Notice that H_λ, when viewed as a function on all of M, has constant orbits along D. We refer to these orbits as *degenerate orbits*.

The Floer cochain complex is defined by

$$\mathrm{CF}^*(X; H_{\lambda,t}) := \bigoplus_{x \in \mathcal{X}(X; H_{\lambda,t})} |\mathfrak{o}_x|_{\mathbf{k}}, \tag{2.6}$$

where $|\mathfrak{o}_x|_{\mathbf{k}}$ is the $\mathbf{k}$-orientation line on the real vector space $\mathfrak{o}_x$ of rank one associated with $x \in \mathcal{X}(X; H_{\lambda,t})$. Fix a choice $J_F : [0,1] \to \mathcal{J}_c(M, D)$. The differential involves the count of maps $u : \mathbb{R} \times S^1 \to X$, asymptotic to $x_\pm$ at $\pm\infty$ satisfying a PDE called *Floer's equation*:

$$\begin{cases} u \colon \mathbb{R} \times S^1 \to X, \\ \lim_{s \to -\infty} u(s, -) = x_0 \\ \lim_{s \to +\infty} u(s, -) = x_1 \\ \partial_s u + J_F(\partial_t u - X_{H_{\lambda,t}}) = 0. \end{cases} \tag{2.7}$$

Denote by

$$\widetilde{\mathcal{M}}(x_0, x_1) \tag{2.8}$$

the moduli space of Floer trajectories between x_0 and x_1, or solutions to (2.7). For generic J_F, (2.8) is a manifold of dimension

$$\deg(x_0) - \deg(x_1)$$

where $\deg(x)$, the index of the local operator D_x associated to x, is equal to $n - CZ(x)$, where $CZ(x)$ is the Conley-Zehnder index of x. The choice of H_λ and J_t near ∞ ensures that an *integrated maximum principle* (Lemma 2.5) holds for solutions to Floer's equation.

The maximum principle ensures that such solutions remain in $X \setminus U_\epsilon$, A suitable version of Gromov compactness then ensures that for generic choices, whenever $\deg(x_0) - \deg(x_1) = 1$, the moduli space $\mathcal{M}(x_0, x_1) := \widetilde{\mathcal{M}}(x_0, x_1)/\mathbb{R}$ is compact of dimension 0. Moreover, orientation theory associates, to every rigid element $u \in \mathcal{M}(x_0, x_1)$ an isomorphism of orientation lines $\mu_u : \mathfrak{o}_{x_1} \to \mathfrak{o}_{x_0}$ and hence an induced map $\mu_u : |\mathfrak{o}_{x_1}|_{\mathbf{k}} \to |\mathfrak{o}_{x_0}|_{\mathbf{k}}$. Using this, one defines the $|\mathfrak{o}_{x_1}|_{\mathbf{k}} - |\mathfrak{o}_{x_0}|_{\mathbf{k}}$ component of the differential

$$(\partial_{CF})_{x_1,x_0} = \sum_{u \in |\mathcal{M}(x_0,x_1)/\mathbb{R}|} \mu_u \tag{2.9}$$

whenever $\deg(x_1) = \deg(x_0) + 1$. In a similar vein, for any $\lambda_1 \geq \lambda_2$, we have continuation maps (canonically defined on the level of cohomology)

$$\kappa_{\lambda_1,\lambda_2} : HF^*(X, H_{\lambda_1,t}) \to HF^*(X, H_{\lambda_2,t})$$

Define

$$SH^*(X) := \varinjlim_{\lambda} HF^*(X, H_{\lambda,t}). \tag{2.10}$$

It is an easy special case of the results of section 4 and the discussion preceeding the proof of Lemma 6.8 of [M] (see also the discussion in Section 10 of [S]) that this agrees with the standard definition of $SH^*(X)$ as defined in [V]. The final bit of structure we will need for the moment is the existence of a product operation on symplectic cohomology. Recall that a *negative cylindrical end*, resp. a *positive cylindrical end* near a puncture x of a Riemann surface Σ consist of a holomorphic map

$$\epsilon_- : (-\infty, 0] \times S^1 \to \Sigma \tag{2.11}$$

resp. a holomorphic map

$$\epsilon_+ : [0, \infty) \times S^1 \to \Sigma \tag{2.12}$$

asymptotic to x. Let Σ be a Riemann surface equipped with suitable cylindrical ends ϵ_i. To each cylindrical end associate a time dependent Hamiltonian H_i. Let $K \in \Omega^1(\Sigma, C^\infty(X))$ be a 1-form on Σ which, along the cylindrical ends, satisfies:

$$\epsilon_i^*(K) = H_i dt$$

whenever $|s|$ is large. To such a K, we may associate a Hamiltonian one form $X_K \in \Omega^1(\Sigma, C^\infty(TX))$ which is characterized by the property that for any tangent vector at a point $z \in \Sigma$, $\vec{r}_z$, we have that $X_K(\vec{r}_z)$ is the Hamiltonian vector-field of $K(\vec{r}_z)$. The most general form of Floer's equation that we will be studying in this paper is:

$$\begin{cases} u \colon \Sigma \to X, \\ (du - X_K)^{0,1} = 0. \end{cases} \tag{2.13}$$

To define the pair of pants product, let Σ be the pair of pants, viewed as a sphere minus three points. Labeling the punctures of Σ by x_1, x_2 and x_0, we equip Σ with positive cylindrical ends around x_1 and x_2 and a negative end around x_0. We fix the following additional data on Σ:

- a (surface-dependent) family of *admissible* J. Further, when restricted to cylindrical ends J should depend only on t.
- a closed 1-form β, which when restricted to the the negative cylindrical end equals $2dt$ and when restricted to the positive cylindrical ends equals dt.
- A perturbation one-form K restricting to $H_{\lambda,t}dt$ on the positive ends and $H_{2\lambda,t}dt$ such that outside of a compact set on X, $X_K = X_{H_{\lambda,t}} \otimes \beta$.

To define the pair of pants product,

$$(- \cdot -) : HF^*(X, H_{\lambda,t}) \otimes HF^*(X, H_{\lambda,t}) \to HF^*(X, H_{2\lambda,t}) \tag{2.14}$$

we count solutions to Equation (2.13) such that

$$\begin{cases} u\colon \Sigma \to X, \\ \lim_{s\to-\infty} u(s,-) = x_0 \\ \lim_{s\to\infty} u(s,-) = x_1 \\ \lim_{s\to\infty} u(s,-) = x_2 \end{cases} \tag{2.15}$$

We finish this section by recording the precise version of the integrated maximum principle that we need. Its proof is a variant of the arguments in [**AS**]. Associated to a solution of Floer's equation (2.13) are two types of energies, the *topological energy*

$$E_{\text{top}}(u) = \int_{\bar{\Sigma}} u^*\omega - d(u^*K) \tag{2.16}$$

and the *geometric energy*

$$E_{\text{geo}}(u) := \frac{1}{2}\int_{\bar{\Sigma}} ||du - X_K|| = E_{\text{top}}(u) + \int_{\bar{\Sigma}} \Omega_K \tag{2.17}$$

where the curvature Ω_K of a perturbation datum K is the exterior derivative of K in the Σ direction.

LEMMA 2.5. *Let $\bar{\Sigma}$ be a compact Riemann surface with boundary and β be a subclosed one form. Let $u\colon \bar{\Sigma} \to U_\epsilon$ be a solution to Floer's equation* (2.13) *with $K = H_\lambda \otimes \beta$ such that $u(\partial\bar{\Sigma}) \subset \partial U_\epsilon$. Then $u(\bar{\Sigma}) \subset \partial U_\epsilon$ and $du - X_H \otimes \beta = 0$.*

PROOF. We compute for such u that the topological energy $E_{\text{top}}(u)$ satisfies

$$E_{\text{top}}(u) = \int_{\bar{\Sigma}} u^*\omega - d(u^*H_\lambda \otimes \beta) \tag{2.18}$$

$$= \int_{\partial\bar{\Sigma}} u^*\hat{\theta} - u^*H_\lambda \otimes \beta \tag{2.19}$$

$$= \int_{\partial\bar{\Sigma}} (u^*\hat{\theta} - \hat{\theta}(X_{H_\lambda})\beta) + \int_{\partial\bar{\Sigma}} \lambda(1-c)\beta \tag{2.20}$$

$$\leq \int_{\partial\bar{\Sigma}} (u^*\hat{\theta} - \hat{\theta}(X_{H_\lambda})\beta) \tag{2.21}$$

$$= -\int_{\partial\bar{\Sigma}} \hat{\theta} \circ J \circ (du - X_{H_\lambda} \otimes \beta) \circ j \tag{2.22}$$

$$= \int_{\partial\bar{\Sigma}} \frac{1-r^2}{r} dr \circ du \circ j, \tag{2.23}$$

Finally, letting $\hat{n}$ denote the outward normal along $\partial^n\Sigma$, observe that $\partial^n\Sigma$ is oriented by the vector $j\hat{n}$. Now we calculate that $dr(du)j(j\hat{n}) = dr(du)(-\hat{n}) \leq 0$, which implies that the final integral, hence $E_{top}(u)$, is non-positive. This implies, because β is subclosed, that $E_{\text{geo}}(u) \leq 0$, so $du = X_K$ everywhere and in particular, r must be constant on $u(\bar{\Sigma})$. □

2.1. Log PSS morphism. We recall the method from [**GP2**] to construct canonical elements in symplectic cohomology. In this section, we suppose that (M, D) is a topological pair. Consider the domain $S = \mathbb{C}P^1 \setminus \{0\}$, thought of as a punctured sphere, with a distinguished marked point $z_0 = \infty$ and a negative cylindrical end near $z = 0$ which for concreteness we take to be given by

$$(s, t) \to e^{s+it}$$

The coordinates (s, t) extend to all of $S \setminus z_0$. Fix a subclosed one-form β which restricts to dt on the cylindrical end and which restricts to zero in a neighborhood of z_0. To be explicit, we consider a non-negative, monotone non-increasing cutoff function $\rho(s)$ such that

$$\rho(s) = \begin{cases} 0 & s \gg 0 \\ 1 & s \ll 0 \end{cases} \tag{2.24}$$

and let

$$\beta = \rho(s)dt. \tag{2.25}$$

Definition 2.6. *For a fixed pair of complex structures $J_0 \in J_c(M, D)$ and J_F, we let $J_S(J_0, J_F)$ denote the space of complex structures:*

$$J_S \in C^\infty(S, J_c(M, D)) \tag{2.26}$$

such that in a neighborhood of z_0, we have that $J_S = J_0$ and such that along the negative strip-like end we have that $J_S = J_F$.

Near z_0, we also fix a distinguished tangent vector which points in the positive real direction. For each $\alpha \in H_*(\bar{X}, \partial\bar{X})$, fix a relative pseudocycle representative [**K**] α_c such that $\partial\alpha_c \subset \partial\bar{X}$. For any such pseudocycle, and orbit x_0 in $\mathcal{X}(M; H_{\lambda,t})$ (possibly degenerate), choose a surface dependent almost-complex structure $J_S \in J_S(J_0, J_F)$.

Definition 2.7. *For any orbit $x_0 \in \mathcal{X}(M; H_{\lambda,t})$, we define $\mathcal{M}_M(x_0)$ as the space of solutions to*

$$u : S \to M$$

satisfying

$$(du - X_{H_{\lambda,t}} \otimes \beta)^{0,1} = 0 \tag{2.27}$$

with asymptotic condition

$$\lim_{s\to-\infty} u(\epsilon(s, t)) = x_0 \tag{2.28}$$

For any $x_0 \in \mathcal{X}(X; H_{\lambda,t})$, consider those $u \in \mathcal{M}_M(x_0)$ such that $u(S) \subset X$ and denote this moduli space by $\mathring{\mathcal{M}}_M(x_0)$. Next form

$$\mathcal{M}(\alpha_c, x_0) := \mathring{\mathcal{M}}_M(x_0) \times_{ev_{z_0}} \alpha_c \tag{2.29}$$

For generic choices of J_S, this is a manifold of dimension $|x_0|-|\alpha|$ provided that $|x_0|-|\alpha| \leq 1$. A standard orientation analysis shows that whenever $|x_0| - |\alpha| = 0$, (rigid) elements $u \in \mathcal{M}(\alpha_c, x_0)$ induce isomorphisms of orientation lines

$$\mu_u : \mathbb{R} \to \mathfrak{o}_{x_0}.$$

Thus, we can define:

$$\mathrm{PSS}_0(P) = \sum_{x_0, |x_0|-|\alpha|=0} \sum_{u \in \mathcal{M}(\alpha_c, x_0)} \mu_u. \tag{2.30}$$

Furthermore, given a generic relative null-bordism Z_b for a pseudo-manifold α_c, we have that

$$\partial_{CF} \circ PSS_0(Z_b) = PSS_0(\alpha_c). \tag{2.31}$$

It is a classical fact that this count gives rise to a well-defined map:

$$PSS_0 : H^*(\bar{X}) \to HF^*(X, H_\lambda) \tag{2.32}$$

The map PSS_{log} is an enhanced version of the classical PSS morphism, involving curves passing through cycles not necessarily in X but at ∞ (D), with various incidence and multiplicity conditions. Let $\mathbf{v}$ be a multiplicity in $\mathbb{Z}^+$.

Definition 2.8. *Fix J_0 , J_F and $J_S \in J_S(J_0, J_F)$. For every orbit $x_0 \in \mathcal{X}(X; H_{\lambda,t})$ and $\mathbf{v}$ as above, define a moduli space*

$$\mathcal{M}(\mathbf{v}, x_0) \tag{2.33}$$

as follows: consider the space of maps

$$u : S \to M$$

satisfying (2.27), (3.18) *and tangency/intersection conditions*

$$u(z) \notin D \text{ for } z \neq z_0; \tag{2.34}$$

$$u(z_0) \text{ intersects } D \text{ with multiplicity } \mathbf{v}\ . \tag{2.35}$$

In the setting of (3.19), the real-projectivized $\mathbf{v}$ normal jets of the map u (with respect to a fixed real tangent ray in $T_{z_0}C$) give an *enhanced evaluation map*

$$\mathrm{Ev}^{\mathbf{v}}_{z_0} : \mathcal{M}(\mathbf{v}, x_0) \to SD. \tag{2.36}$$

For each $\alpha \in H^*(SD)$, fix a pseudocycle representative α_c.

Definition 2.9. *The moduli space $\mathcal{M}(\mathbf{v}, \alpha_c, x_0)$ is defined to be the moduli space*

$$\mathcal{M}(\mathbf{v}, x_0) \times_{\mathrm{Ev}^{\mathbf{v}}_{z_0}} \alpha_c, \tag{2.37}$$

As is shown in Lemma 4.10 of [**GP2**], the virtual dimension of $\mathcal{M}(\mathbf{v}, \alpha_c, x_0)$ is given by

$$\mathrm{vdim}(\mathcal{M}(\mathbf{v}, \alpha_c, x_0)) = |x_0| - |\alpha t^{\mathbf{v}}|, \tag{2.38}$$

For a generic choice of almost complex structure and when $\mathrm{vdim}(\mathcal{M}(\mathbf{v}, \alpha_c, x_0)) \leq 1$, the moduli space is a manifold of the expected dimension by standard arguments(see Lemma 4.16 of [**GP2**], which is in turn an adaptation of Lemmas 6.5-6.6 of [**CM**]). For $\lambda > \mathbf{v}$, we define:

$$\mathrm{PSS}^{\lambda}_{log}(\alpha_c t^{\mathbf{v}}) = \sum_{x_0, \mathrm{vdim}(\mathcal{M}(\mathbf{v}, \alpha_c, x_0))=0} \sum_{u \in \mathcal{M}(\mathbf{v}, \alpha_c, x_0)} \mu_u \in HF^*(X, H_{\lambda,t}) \tag{2.39}$$

where once more, for a rigid element $u \in \mathcal{M}(\mathbf{v}, \alpha_c, x_0)$, $\mu_u : \mathbb{R} \to \mathfrak{o}_{x_0}$ is the isomorphism induced on orientation lines (and by abuse of notation, their $\mathbf{k}$-normalizations) by the gluing theory. The key point is the following, which is a small modification of Lemma 4.13 of [**GP2**](see also Lemma 7.1 of [**S**])

LEMMA 2.10. *We have that* $\partial_{CF} \circ \mathrm{PSS}^{\lambda}_{log}(\alpha_c t^{\mathbf{v}}) = 0$

PROOF. To prove this, we need to show that, when the dimension is one, the moduli space $\mathcal{M}(\mathbf{v}, \alpha_c, x_0)$ admits a compactification (in the sense of Gromov-Floer convergence) $\overline{\mathcal{M}}(\mathbf{v}, \alpha_c, x_0)$, such that:

$$\partial\overline{\mathcal{M}}(\mathbf{v}, \alpha_c, x_0) = \bigsqcup_{x, |x_0| - |x'| = 1} \mathcal{M}(\mathbf{v}, \alpha_c, x') \times \mathcal{M}(x_0, x')$$

To do this, because M is compact, we may apply standard Gromov-Floer compactness. As there is no sphere bubbling, we only have to rule out PSS solutions breaking along a degenerate orbit y. Then have that the limiting u breaks into two curves (u_1, u_2) where u_1 is an element of $\mathcal{M}_M(y)$ and u_2 is a non-trivial Floer trajectory in $\overline{\mathcal{M}}(x_0, y)$. To rule this out, observe that by the maximum principle, we must have that u_1 lies entirely in D. In particular, we observe that we must have

$$\int_S u_1^*\omega \geq 0. \tag{2.40}$$

We now show that u_2 cannot exist by energy considerations. Namely, let r be the bundle coordinate on U and we consider the slice where $r = \epsilon$ as before. Along this slice, recall that we have that

$$H = \lambda(c - r^2) \tag{2.41}$$

Consider the piece of this Floer trajectory, which lives above $r = \epsilon$. We have that the energy of $\bar{S} = u^{-1}(U_\epsilon)$ of this piece of the curve for the symplectic form ω can be estimated by

$$E(\bar{S}) \leq \mathbf{v} - c\lambda + \int_{\partial\bar{S}} (u^*\hat{\theta} - (H\beta)) \tag{2.42}$$

$$= \mathbf{v} - c\lambda + \int_{\partial\bar{S}} u^*\hat{\theta} - \hat{\theta}(X)\beta + \int_{\partial\bar{S}} (\lambda(1-c)\beta) \tag{2.43}$$

$$\leq \int_{\partial\bar{S}} u^*\hat{\theta} - \hat{\theta}(X)\beta \tag{2.44}$$

And the rest proceeds as we have seen previously. As a result we have that the moduli space is compact. □

For any $\lambda > 0$, set

$$H^{\leq\lambda}_{log}(M, D) := H^*(X) \bigoplus_{\mathbf{v} \leq \lambda} H^*(SD)t^{\mathbf{v}} \tag{2.45}$$

Parallel to Lemma 4.26 of [**GP2**], the formula (2.39) is independent of the choice of pseudocycle, giving rise to a well-defined map

$$\mathrm{PSS}^{\lambda}_{log} : H^{\leq\lambda}_{log}(M, D) \to HF^*(X, H_{\lambda,t}) \tag{2.46}$$

Moreover, this map is compatible with continuation maps by the argument of Lemma 4.24 of [**GP2**], giving rise to a map

$$\mathrm{PSS}_{log} : H^*_{log}(M, D) \to SH^*(X) \tag{2.47}$$

2.2. PSS is a map of rings. To define the ring structure on $H^*_{log}(M, D)$, first observe that the identification ψ gives rise to a natural identification $\psi_b : SD \cong \partial \bar{X}$.

DEFINITION 2.11. *We define the ring structure on $H^*_{log}(M, D)$ to be the unique (graded-) commutative ring structure such that*

- *The natural inclusion $H^*(X) \to H^*_{log}(M, D)$ is ring homomorphism.*
- *For $\alpha_1 \in H^*(X)$, $\alpha_2 \in H^*(SD)$ and $\mathbf{v} \neq 0$, we have*

$$\alpha_1 \cdot \alpha_2 t^{\mathbf{v}} = (\psi_b^*(\alpha_1) \cup \alpha_2) t^{\mathbf{v}} \tag{2.48}$$

- *Finally, for $\alpha_1, \alpha_2 \in H^*(SD)$*

$$\alpha_1 t \cdot \alpha_2 t = (\alpha_1 \cup \alpha_2) t^2 \tag{2.49}$$

- *For $\alpha \in H^*(SD)$, and $\mathbf{v}$ not equal to zero, then*

$$[SD]t \cdot \alpha_1 t^{\mathbf{v}} = \alpha_1 t^{\mathbf{v}+1} \tag{2.50}$$

The third and fourth bulleted relations above can be replaced by the slightly cleaner relation that for $\alpha_1 t^{\mathbf{v}_1}, \alpha_2 t^{\mathbf{v}_2}$ with both $\mathbf{v}_1, \mathbf{v}_2 \neq 0$, then

$$\alpha_1 t^{\mathbf{v}_1} \cdot \alpha_2 t^{\mathbf{v}_2} = \alpha_1 \cup \alpha_2 t^{\mathbf{v}_1 + \mathbf{v}_2} \tag{2.51}$$

However, the presentation we give is slightly simpler for comparing ring structures to symplectic cohomology. The argument that the PSS map defined above preserves the ring structure follows a standard pattern in TQFT. Namely, we work over a parameter space $q \in (0, b]$ with $b > 0$ close to zero. We again consider the surface $S_{q,2} = \mathbb{C}P^1 \setminus \{0\}$, with a negative cylindrical end as before, but this time with two distinguished marked points at $z_1 = -1/q$ and $z_2 = \infty$. For this argument it will also be convenient to set $w = z^{-1}$. We consider a subclosed one form β which satisfies

- The form β restricts to $2dt$ on the cylindrical end
- $\beta = 0$ in balls surrounding $w(z_1) = -q$ and $w(z_2) = 0$.

We equip each of these points asymptotic markers, in the negative real direction at z_1 and in the positive real direction at z_2. Next, define a moduli space $\mathcal{M}_q(\mathbf{v}_1, \mathbf{v}_2; x_0)$ of pairs (q, u), with $q \in (0, b]$ and

$$u : S_{q,2} \to M$$

as usual satisfying

$$(du - X_{H_{\lambda,t}} \otimes \beta)^{0,1} = 0 \tag{2.52}$$

with asymptotic condition

$$\lim_{s \to -\infty} u(\epsilon(s, t)) = x_0 \tag{2.53}$$

and tangency/intersection conditions

$$u(x) \notin D \text{ for } x \neq z_i; \tag{2.54}$$

$$u(z_1) \text{ intersects } D \text{ with multiplicity } \mathbf{v}_1. \tag{2.55}$$

$$u(z_2) \text{ intersects } D \text{ with multiplicity } \mathbf{v}_2\ . \tag{2.56}$$

Where we understand that $\mathbf{v}_i = 0$ means that the curve does not pass through D at z_i. If one of the $\mathbf{v}_i = 0$, we choose pseudocycles α_i representing classes in $H_*(\bar{X}, \partial\bar{X})$ and extend these $\mathbb{R}^*$ invariantly over $X \setminus \bar{X} \cong \mathring{U}_\delta \setminus D$. Otherwise we simply take a α_i to be a pseudocycle on SD.

Theorem 2.12. *We have that*

$$\mathrm{PSS}_{log}(\alpha_1 t^{\mathbf{v}_1} \cdot \alpha_2 t^{\mathbf{v}_2}) = \mathrm{PSS}_{log}(\alpha_1 t^{\mathbf{v}_1}) \cdot \mathrm{PSS}_{log}(\alpha_2 t^{\mathbf{v}_2}) \tag{2.57}$$

Proof. In view of the defining relations given in Definition 2.11, it suffices to consider the cases where

(1) $\mathbf{v}_1 = 0$
(2) $\mathbf{v}_1 = \mathbf{v}_2 = 1$
(3) $\mathbf{v}_1 = 1$, α_1 is the fundamental cycle on SD.

Consider the moduli spaces

$$\mathcal{M}_q(\mathbf{v}_1, \mathbf{v}_2; \alpha_1, \alpha_2; x_0) := \mathcal{M}_q(\mathbf{v}_1, \mathbf{v}_2; x_0) \times_{\mathrm{Ev}} \alpha_1 \times_{\mathrm{Ev}} \alpha_2 \tag{2.58}$$

which for generic choices is a moduli space of the expected dimension. Suppose first that we are in case (1) above with $\mathbf{v}_1 = 0$. If $\mathbf{v}_2 = 0$ it is well known that the classical PSS morphism is a ring map, so assume that $\mathbf{v}_2 > 0$. When the moduli space (2.58) has dimension one, we claim that it may be completed over $q \to 0$ to a one dimensional manifold with boundary whose fiber over $q = 0$ is given by

$$\overline{\mathcal{M}}(\mathbf{v}_1 + \mathbf{v}_2, \psi_b^{-1}(\alpha_1 \cap \partial X) \cap \alpha_2, x_0) \tag{2.59}$$

To see, this consider a sequence of curves with u_q, $q \to 0$. Since there is no bubbling, the limiting is a curve $u_0 : S \to M$ which intersects D at exactly one point $u_0(z_0)$ with a constant sphere bubble glued on at z_0. Choose a small open set W of D about $u_0(z_0)$ together with a trivalization

$$\ell : W \times \mathbb{C} \cong ND_{|W} \tag{2.60}$$

Let U_W be $UD_\epsilon \cap \pi^{-1}(W)$. Identifying U_W with its image under ψ, we have that there is an open set $U_S \subset S$ with $w = -q, 0 \subset U_S$ such that $u_q(U_S) \subset U_W$ for q sufficiently small. In view of (2.60), we have that U_W comes equipped with a projection $\pi_{\mathbb{C}} : U_W \to \mathbb{C}$. Consider the maps $\bar{u}_q = \pi_{\mathbb{C}} \circ u_q : U_S \to \mathbb{C}$. We have

$$\bar{u}_q = a_q(w)^{\mathbf{v}_2} + O(|w|^{\mathbf{v}_2+1}) \tag{2.61}$$

by Lemma 3.4 of [IP] with $a_q \neq 0$. Moreover, we have that

$$\mathrm{Ev}_{z_2}(u_q) = ([a_q], \pi \circ u(0)) \in \alpha_2 \tag{2.62}$$

Writing $u_q(z_1) = (\bar{u}_q(z_1), \pi \circ u(z_1))$, we have that

$$u_q(z_1) = (a_q q^{\mathbf{v}_2} + O(q^{\mathbf{v}_2+1}), \pi \circ u_q(z_1)) \in \alpha_1 \tag{2.63}$$

Because there is no bubbling, C^∞ convergence of the maps u_q imply that that $a_q \to a_0$ and $\lim_{q\to 0} \bar{u}(z_2)/q^{\mathbf{v}_2} \to a_0$. It follows that

$$\mathrm{Ev}_{z_0} = ([a_0], \pi \circ u(0)) \in \psi_b^{-1}(\alpha_1 \cap \partial X) \cap \alpha_2 \tag{2.64}$$

To go the other direction is an elementary gluing result which follows the arguments of Chapter 10 of [**MS**] quite closely and which we regard as standard. It remains to consider the remaining cases when $\mathbf{v}_1 = 1$, and either $\mathbf{v}_2 = 1$ or $\alpha_1 = [SD]$. Let us consider case (2) first. In the limit, we have that $\pi \circ u(w) = b_0 w^2 + O(|w|^3)$. Comparing with (2.61), C^∞ convergence together with the fact that $\pi \circ u_q(-q) = 0$ implies that

$$-a_q q + (b_0 + \epsilon_q)q^2 + O(q^3) = 0 \tag{2.65}$$

with $\epsilon_q \to 0$. It therefore follows that $a_q/q \to b_0$. Performing a Taylor expansion about z_1 instead and writing $u_q(w) = \tilde{a}_q(w+q) + O(|w+q|^2)$, the same reasoning shows that $\tilde{a}_q/q \to -b_0$. The curves therefore limit to maps in

$$\mathcal{M}(2, \alpha_1 \cap \alpha_2, x_0) \tag{2.66}$$

The same analysis also shows that in case (3), as $q \to 0$, the curves limit to maps in

$$\mathcal{M}(1 + \mathbf{v}_2, \alpha_2, x_0) \tag{2.67}$$

In each of the cases (2.59), (2.66), (2.67), the counting of curves in these limits by definition defines the composition

$$\mathrm{PSS}(\alpha_1 t^{\mathbf{v}_1} \cdot \alpha_2 t^{\mathbf{v}_2}) \tag{2.68}$$

Next consider the moduli space $\mathcal{M}_b(M, \mathbf{v}_1, \mathbf{v}_2; \alpha_1, \alpha_2; x_0)$ which is the restriction of the above moduli space to domains $S_{b,2}$. We have that the operation defined by $\mathcal{M}_b(M, \mathbf{v}_1, \mathbf{v}_2; \alpha_1, \alpha_2; x_0)$ is homotopic to (2.68). Let Σ denote the pair of pants, with three standard cylindrical ends attached as in (2.15). We consider the nodal domain S_n of the form

$$S \cup_\epsilon \Sigma \cup_\epsilon S$$

where the negative strip like ends of S are glued to the positive strip like ends of Σ, a pair of pants. Maps from $S_n \to M$ are given by the fiber product of moduli spaces given by:

$$\coprod_{x_1, x_2} \mathcal{M}(\mathbf{v}_1, \alpha_1, x_1) \times \mathcal{M}(\Sigma, x_0, x_1, x_2) \times \mathcal{M}(\mathbf{v}_2, \alpha_2, x_2)$$

We construct a homotopy between this moduli space and $\mathcal{M}_b(M, \mathbf{v}_1, \mathbf{v}_2; x_0)$ in two steps. First, we perform a finite connect sum along the cylindrical ends. Then, we can further homotopy the complex structure and Floer datum to the domain $S_{b,2}$ above. We thus reach the desired conclusion. □

3. PSS is an isomorphism

3.1. Constraining low energy solutions and the "low energy" PSS map. In this subsection, we will prove geometric constraints for certain PSS solutions under the assumption that τ from (2.5) is small and J_S is sufficiently close to being split inside of U. Our strategy will be to first consider the limiting case where $\tau = 0$ and J_S is split inside of U and then to apply Gromov compactness. Let $g(r^2)$ be a function which agrees with r^2 iff

$r \leq \delta$ and which agrees with zero when $r > 2\delta$ and such that $\frac{d}{dr}(r^2 - g(r^2)) \geq 0$. Consider the semi-symplectic form

$$\omega_{\mathrm{red}} = \omega - d(g(r)\theta) \tag{3.1}$$

If $\tau = 0$ and J_S is taken split, then for any PSS solution $u : S \to M$, vector $Y \in Hom(TS, u^*TM)$, and point $s \in S$ with local coordinate system $s + it$, we can mimic the usual construction of geometric energy by defining:

$$||Y||^2_{\mathrm{red}} = |Y_s|^2 + |Y_t|^2 \tag{3.2}$$

where on the right hand side of the equation, the notation $|\cdot|^2$ means $\omega_{\mathrm{red}}(\cdot, J_S \cdot)$. Let $E_{\mathrm{red}}(u)$ denote the geometric energy with respect to the semi-symplectic form given by

$$E_{\mathrm{red}}(u) = \int_S ||du - X_H \otimes \beta||^2_{\mathrm{red}} vol_S \tag{3.3}$$

where $vol_S = ds \wedge dt$. It is a standard calculation that the expression inside the integrand is globally well-defined and, in fact, because we have that $\omega_{\mathrm{red}}(X_{H_\lambda}, -) = 0$ one obtains

$$E_{\mathrm{red}}(u) = \int_S u^*(\omega_{\mathrm{red}}) \tag{3.4}$$

For any $x_0 \in \mathcal{X}_d(X, H_{\lambda,t})$ with $d > 0$, let $A(x_0) \in H_2(M, x_0)$ denote the relative homology class of the natural fiber capping disc.

Lemma 3.1. *Fix $\mathbf{v} > 0$ and α_c and assume that J_S is a split almost complex structure in U. Assume that $x_0 \in \mathcal{X}_{\mathbf{v}}(X, H_\lambda)$.*

- *For any $u \in \mathcal{M}(\mathbf{v}, \alpha_c, x_0)$, we have that $u(S) \subset U_\delta$ for any $u \in \mathcal{M}(\mathbf{v}, \alpha_c, x_0)$ and u projects trivially to D.*
- *For any $d > \mathbf{v}$, we have that for any $x_0 \in \mathcal{X}_d(X, H_\lambda)$, $\mathcal{M}(\mathbf{v}, \alpha_c, x_0) = \emptyset$.*

Proof. The energy $E_{\mathrm{red}}(u)$ is non-negative and strictly positive if u does not lie strictly in a fiber. Any solution $u \in \mathcal{M}(\mathbf{v}, \alpha_c, x_{\mathbf{v}})$ for $x_{\mathbf{v}} \in \mathcal{X}_{\mathbf{v}}(X, H_\lambda)$ satisfies $E_{\mathrm{red}}(u) = 0$ and hence must lie in U_δ and project trivially to D. The argument for the second claim is the same. □

Similarly, we have that under the same assumptions, for any pair $x_0 \in \mathcal{X}_{d'}(X, H_\lambda)$ and $x_1 \in \mathcal{X}_d(X, H_\lambda)$, with $d' > d$ we have that $\mathcal{M}(x_0, x_1)$ is empty.

Lemma 3.2. *Fix $\mathbf{v} > 0$ and α_c and in addition a complex structure $J_{S,0}$ which is split inside of U. There exists a τ_0 such that for $\tau < \tau_0$ and for any J_S, which satisfies*

- *$|J_S - J_{S,0}|_{C^\epsilon} < \tau_0$*
- *J_S is split in a small neighborhood of $r = \delta$ (as well as by definition in U_ϵ)*

we have that:

- *For $d > \mathbf{v}$ and $x_0 \in \mathcal{X}_d(X, H_{\lambda,t})$, we have that $\mathcal{M}(\mathbf{v}, \alpha_c, x_0) = \emptyset$.*
- *Assume that $x_0 \in \mathcal{X}_{\mathbf{v}}(X, H_{\lambda,t})$. Consider the moduli space $\mathcal{M}(\mathbf{v}, \alpha_c, x_0)$, $u(S) \subset U_\delta$ for any $u \in \mathcal{M}(\mathbf{v}, \alpha_c, x_0)$ and the relative homology class $[u] = A(x_0)$.*

Proof. This is essentially a standard argument using a Morse Bott version of Gromov-Floer compactness. The one slight complication is that as $\tau \to 0$ we do not have Morse-Bott orbits near the boundary of U and hence cannot guarantee the unique existence of limits for Floer solutions. However, suppose a solution spends arbitrarily long intervals of s close to an orbit near the boundary of U, the positive end of such a PSS solution limits to some orbit in $\mathcal{X}_d(X, H_{\lambda,t})$. The integrated maximum principle shows that such a solution cannot exist. □

Corollary 3.3. *Fix $J_{S,0}$ (this by definition includes a corresponding $J_{F,0}$) which is split in U and take τ sufficiently small and J_S sufficiently close to $J_{S,0}$ so that 3.2 holds. The Floer complex of $\mathrm{CF}^*(X, H_{\lambda,t})$ admits a filtration by the multiplicity d, $H^*_{log}(M,D)^{\leq \lambda}$ is triviallly filtered by $\mathbf{v}$ and the PSS_{log} map respects this filtration.*

Proof. This follows immediately from Lemma 3.2 when $d \geq 1$ and follows from the integrated maximum principle when $d = 0$. □

Our argument will proceed by analyzing the "diagonal terms" of the PSS map with respect to the filtrations obtained in 3.3. More precisely, let

$$\mathrm{CF}^*(X; H_{\lambda,t})_{(d)} := \bigoplus_{x \in \mathcal{X}_d(X; H_{\lambda,t})} |\mathfrak{o}_x|_{\mathbf{k}}, \tag{3.5}$$

After choosing J_F as in Corollary 3.3, we may equip this complex with a differential $\partial_{CF_{(d)}}$ which counts Floer trajectories only between orbits in $\mathcal{X}_d(X; H_{\lambda,t})$. We denote the cohomology of this complex by $HF^*(X, H_{\lambda,t})_{(d)}$. Similarly, we let

$$H^*_{log}(M,D)_{(d)} := H^*_{log}(M,D)^{\leq d}/H^*_{log}(M,D)^{\leq d-1} \tag{3.6}$$

By counting PSS solutions in the homology classes $A(x_0)$, we obtain for every $d < \lambda$, a map

$$\mathrm{PSS}^{(d)}_{log} : H^*_{log}(M,D)_{(d)} \to HF^*(X, H_{\lambda,t})_{(d)} \tag{3.7}$$

Standard Morse-Bott analysis going back to Floer shows that when $d = 0$, we have a canonical identification

$$I_F : HF^*(X, H_{\lambda,t})_{(d)} \cong H^*(X) \tag{3.8}$$

and when $d > 0$, we have an identification

$$I_F : HF^*(X, H_{\lambda,t})_{(d)} \cong H^*(SD) \tag{3.9}$$

Lemma 3.4. *The map (2.47) is an isomorphism if for every λ, (3.7) is an isomorphism for each $d < \lambda$.*

Proof. The spectral sequence associated to the filtration by d is bounded below and exhaustive. Thus to prove that (2.46) is an isomorphism for every λ, it suffices to prove that it is an isomorphism on the first page. This is by definition equivalent to (3.7) being an isomorphism. After passing to direct limits, we obtain that (2.47) is an isomorphism. □

3.2. A low energy inverse. We will now construct a one-sided inverse to (3.7) for $d \geq 1$ (the case with $d = 0$ is classical). Consider the standard projective bundle $PD = P(ND \oplus \mathcal{O}_D)$ over D. Let $\pi_P : PD \to D$ denote the standard projection to D. There are two natural holomorphic sections D_0 and D_∞ and we may algebraically identify

$$PD \setminus D_\infty = ND = Tot(\mathcal{O}(D)) \tag{3.10}$$

$$PD \setminus D_0 \cong Tot(\mathcal{O}(-D)) \tag{3.11}$$

It is also worth remarking that

$$PD \setminus (D_\infty \cup D_0) = ND \setminus D = Tot(\mathcal{O}(D)) \setminus D \cong SD \times \mathbb{R}$$

where the last isomorphism makes sense in the smooth category only. Turning to symplectic forms, we equip PD with the standard form

$$\omega_{PD} = 2\delta d(\frac{2p}{1+p^2}\theta) + \pi_P^*(\omega_D) \tag{3.12}$$

A neighborhood of U_{D_0} in the projective bundle can be identified with a neighborhood of ND symplectically by setting

$$1/2r_{D_0}^2 = \frac{2p}{1+p^2} \tag{3.13}$$

Symmetrically, we can identify a neighborhood of D_∞ inside of PD with a disc bundle and the projective bundle as arising from a symplectic sum construction of these two disc bundles. We may embed $U_\delta \subset U_{D_0} \subset PD$. Notice that we can take the functions $H_{\lambda|U_\delta}$ and extend them by zero to obtain functions $H_\lambda^{loc} : PD \to \mathbb{R}$. We may perturb these functions inside of U_d, $d \geq 1$ as in (2.5) to obtain functions $H_{\lambda,t}^{loc}$. By abuse of notation, we again label non-constant orbits by $\mathcal{X}_d(X^{loc}; H_{\lambda,t}^{loc}) = \mathcal{X}_d(X; H_{\lambda,t})$. Set X^{loc} to be the open set $PD \setminus D_0$.

We will need to adapt the Floer theoretic structures introduced in the previous sections to this local setting.

Definition 3.5. *Let $J_c(PD, \mathbf{D})$ denote the space of complex structures which are split inside of U_ϵ and some small neighborhood of D_∞. We let J_F^{loc} denote the space of maps $[0,1] \to J_c(PD, \mathbf{D})$ which are time independent in a neighborhood of D_∞.*

We can then consider Floer's equation for these Hamiltonians:

$$\begin{cases} u\colon \mathbb{R} \times S^1 \to X^{loc}, \\ \partial_s u + J_F^{loc}(\partial_t u - X_{H_{\lambda,t}^{loc}}) = 0. \end{cases} \tag{3.14}$$

subject to the usual asymptotic constraints. For any two orbits x_0, x_1, let $\mathcal{M}(X^{loc}; x_0, x_1)$ denote the moduli space of these solutions modulo $\mathbb{R}$-translations. By a similar Morse-Bott compactness argument, we have that:

Lemma 3.6. *Fix τ sufficiently small and J_F^{loc} sufficiently close to a split time-dependent almost complex structure. For any two orbits $x_0 \in \mathcal{X}_d(X^{loc}; H_{\lambda,t}^{loc})$ and $x_1 \in \mathcal{X}_{d'}(X^{loc}; H_{\lambda,t}^{loc})$, we have that the moduli space of Floer trajectories $\mathcal{M}(X^{loc}; x_0, x_1)$ in X^{loc} is empty for $d > d'$, and when $d = d'$, we have a bijection between $\mathcal{M}(X^{loc}; x_0, x_1)$ and $\mathcal{M}(x_0, x_1)$.*

As in (3.5) set:

$$\mathrm{CF}^*(X^{loc}; H_{\lambda,t}^{loc})_{(d)} := \bigoplus_{x \in \mathcal{X}_d(X^{loc}; H_{\lambda,t}^{loc})} |\mathfrak{o}_x|_{\mathbf{k}}, \tag{3.15}$$

and after choosing J_F^{loc} as in Lemma 3.6, define a differential counting (as usual with appropriate signs) elements $u \in \mathcal{M}(X^{loc}; x_0, x_1)$. After choosing τ sufficiently small and J_F^{loc} sufficiently close to a split time-dependent almost complex structure, we have a canonical identification of cohomologies:

$$HF^*(X, H_{\lambda,t})_{(d)} \cong HF^*(X^{loc}, H_{\lambda,t}^{loc})_{(d)} \tag{3.16}$$

Definition 3.7. *Given $x_0 \in \mathfrak{X}_d(X^{loc}; H^{loc}_{\lambda,t})$ and $\alpha_c \in SD_0$, we may choose $J^{loc}_S \in C^\infty(S, J_c(PD, \mathbf{D}))$ on S which agrees with J^{loc}_F along the ends and with some split J_0 near $z = \infty$. Let $\mathcal{M}_{PSS}(X^{loc}, x_0)$ denote the moduli space of solutions $u : S \to PD$ satisfying*

$$(du - X_{H^{loc}_{\lambda,t}} \otimes \beta)^{0,1} = 0 \tag{3.17}$$

with asymptotic condition

$$\lim_{s \to -\infty} u(\epsilon(s,t)) = x_0 \tag{3.18}$$

and tangency/intersection conditions

$$u(z) \notin D_0 \text{ for } z \neq z_0; \tag{3.19}$$

$$u(z_0) \text{ intersects } D_0 \text{ with multiplicity } d\ . \tag{3.20}$$

Consider the fiber product moduli space:

$$\mathcal{M}_{PSS}(X^{loc}, \alpha_c, x_0) := \mathcal{M}_{PSS}(X^{loc}, x_0) \times_{\mathrm{Ev}_{z_0}} \alpha_c \tag{3.21}$$

After choosing τ sufficiently small and J^{loc}_S sufficiently close to split, the resulting map $\mathrm{PSS}^{(d)}_{loc} : H^*_{log}(M, D)_{(d)} \to HF^*(X^{loc}, H^{loc}_{\lambda,t})_{(d)}$ agrees with (3.7) after making the identification (3.16). We will need one final moduli space, which has not appeared in our discussion before.

Definition 3.8. *Consider the "thimble domain" $S^\vee := \mathbb{R} \times S^1 \cup \{0\}$. For $x_0 \in \mathfrak{X}_d(X^{loc}; H_{\lambda,t})$, we let $\widetilde{\mathcal{M}}_{SSP}(\mathbf{v}, X^{loc}, x_0)$ denote the moduli space of solutions to Floer's equation (3.14) which satisfy*

$$u \cdot D_\infty = \mathbf{v} \cdot \{0\} \tag{3.22}$$

$$\lim_{s \to +\infty} u(s, -) = x_0. \tag{3.23}$$

We let $\mathcal{M}_{SSP}(\mathbf{v}, X^{loc}, x_0)$ denote the quotient of this moduli space by $\mathbb{R}$ translations.

As usual, these moduli spaces have Gromov compactifications $\overline{\mathcal{M}}_{SSP}(\mathbf{v}, X^{loc}, x_0)$ (one could also consider SFT refinements of these compactifications, but this is unnecessary for our purposes). We will let $A^\vee(x_0)$ denote the relative homology class of the fiber capping disc for x_0 in X^{loc}. We will use the simpler notation $\mathcal{M}_{SSP}(x_0)$ for the moduli space of SSP thimbles in relative homology class $A^\vee(x_0)$.

Lemma 3.9. *For τ sufficiently small, $J_{F,loc}$ close to split and any orbit $x_0 \in \mathfrak{X}_d(X^{loc}; H^{loc}_{\lambda,t})$, let $u \in \overline{\mathcal{M}}_{SSP}(\mathbf{v}, X^{loc}, x_0)$. Then $[u] = B \# A^\vee(x_0) \in H_2(PD, x_0)$ for B with $\pi^*(\omega_D)(B) \geq 0$. In particular $\mathbf{v} \leq d$.*

Choose τ small enough and $J_{F,loc}$ sufficiently close to split so that Lemmas 3.6 and 3.9 hold. As with the PSS map, we have an enhanced evaluation map

$$Ev_\infty : \mathcal{M}_{SSP}(x_0) \to SD_\infty \tag{3.24}$$

For a generic choices, $\mathcal{M}_{SSP}(x_0)$ has dimension

$$\mathrm{vdim}(\mathcal{M}_{SSP}(x_0)) = 2n - 1 + 2\mathbf{v}(m-1) - |x_0|. \tag{3.25}$$

For any $\beta_c \to SD_\infty$ such that $|\beta_c| = \mathrm{vdim}(\mathcal{M}_{SSP}(x_0))$, we have that for generic choices, the moduli space

$$\mathcal{M}_{SSP}(x_0, \beta_c) := \mathcal{M}_{SSP}(x_0) \times_{\mathrm{Ev}_\infty} \beta_c \tag{3.26}$$

is a zero dimensional manifold. Choose a basis of pseudocycles for $H_*(SD_\infty, \mathbf{k})$, β_i over $\mathbf{k}$. For $\mathcal{X}_d(X^{loc}; H^{loc}_{\lambda,t})$, define

$$\mathrm{SSP}^{(d)}_{log}(\mathfrak{o}_{x_0}) = \sum_{\beta_i, \mathrm{vdim}(\mathcal{M}_{SSP}(x_0,\beta_i))=0} \sum_u \mu(u)\beta_i^\vee t^d \tag{3.27}$$

Lemma 3.10. *For any class $\alpha_c t^{\mathbf{v}} \in H^*_{log}(M, D)$, we have that*

$$\mathrm{SSP}^{(\mathbf{v})}_{log} \circ \mathrm{PSS}^{(\mathbf{v})}_{log}(\alpha_c t^{\mathbf{v}}) = \alpha_c t^{\mathbf{v}} \tag{3.28}$$

Proof. Consider the moduli space of spheres with two marked points at $z_0 = \infty$ and $z_1 = 0$ with Floer data coming from the natural gluing of the Floer data defining the moduli spaces $\mathcal{M}_{PSS}(\alpha_c, x_0)$ and $\mathcal{M}_{SSP}(x_0)$. We assume that J_S is taken surface-independent in a neighborhood of D_∞. We further require that

$$\begin{cases} u^{-1}(D_0) = \mathbf{v} \cdot z_0 \\ u^{-1}(D_\infty) = \mathbf{v} \cdot z_1 \end{cases} \tag{3.29}$$

Denote this moduli space by $\mathcal{M}_{0,2}(\mathbf{v}, X^{loc})$. We will consider the moduli spaces

$$\mathcal{M}_{0,2}(\mathbf{v}, \alpha_c, \beta_i) := \alpha_c \times_{Ev_{z_0}} \mathcal{M}_{0,2}(\mathbf{v}, X^{loc}) \times_{Ev_\infty} \beta_i \tag{3.30}$$

For generic choices of J_S and β_i, this is a manifold of the correct dimension. We will be interested in cases when $\mathcal{M}_{0,2}(\mathbf{v}, \alpha_c, \beta_i)$ has dimension one. In this case, the Gromov compactification of $\mathcal{M}_{0,2}(\mathbf{v}, \alpha_c, \beta_i)$ has two strata $\partial_B\overline{\mathcal{M}}_{0,2}$ and $\partial_S\overline{\mathcal{M}}_{0,2}$. We have that

$$\partial_B\overline{\mathcal{M}}_{0,2} := \coprod_{x_0, \mathrm{vdim}(\mathcal{M}_{PSS}(\alpha_c, x_0)=0)} \mathcal{M}_{PSS}(\alpha_c, x_0) \times \mathcal{M}_{SSP}(x_0, \beta_i)$$

and $\partial_S\overline{\mathcal{M}}_{0,2}$ is the moduli space of J_0-holomorphic spheres with two marked points satisfying (3.29) modulo $\mathbb{R}$-translation in the domain with $Ev_{z_0}(u) \subset \alpha_c$ and $Ev_\infty(u) \subset \beta_i$. The evaluation of $\partial_B\overline{\mathcal{M}}_{0,2}$ gives rise to the left-hand side of (3.28) and the evaluation of $\partial_S\overline{\mathcal{M}}_{0,2}$ gives rise to the right hand side of (3.28). □

Theorem 3.11. *The map (3.7) is an isomorphism.*

Proof. In view of (3.9), we see that to prove that (3.7) is an isomorphism, it suffices to prove that the map is injective. The fact that is injective follows from Lemma 3.10. □

3.3. Additional results. Throughout this subsection we will assume that the pair (M, D) is log-Calabi-Yau, i.e. $m = 1$. We may remark that in the log Calabi-Yau case, we may assume that the grading of all orbits obtained by the perturbation (2.5) is ≤ 0 and that there is a unique orbit of grading 0 in both $\mathcal{X}_0(X; H_{\lambda,t})$ and $\mathcal{X}_d(X; H_{\lambda,t})$. In the second case, we denote this unique orbit by x_d. Let α_c be the fundamental pseudo-cycle of SD, we have the following lemma which is a mild modification of Lemma 2.10.

Lemma 3.12. *For any $\mathbf{v} \geq 1$, let α_c denote the fundamental class of SD. The element $\mathrm{PSS}_{log}(\alpha_c t^{\mathbf{v}})$ defines a class in symplectic cohomology $SH^*(X)$.*

Proof. The difference between this situation and Lemma 2.10 is that in the present situation sphere bubbling can arise near the point z_0. However, in this case, after passing to the somewhere injective images of curves, this sphere bubbling occurs in codimension two (this works with either the standard Deligne-Mumford compactification or SFT style enhancements). ☐

As a result, we obtain a map

$$\mathrm{PSS}_{log} : H^0_{log}(M, D) \to SH^0(X) \tag{3.31}$$

Theorem 3.13. *The map* (3.31) *is an isomorphism.*

Proof. The arguments of subsections 3.1 and 3.2 apply without change when restricted to the degree zero pieces. ☐

Let $\alpha_{\mathbf{v}}$ denote a copy of the fundamental class on either X or SD for each $\mathbf{v}$. Denote the resulting elements $\alpha_{\mathbf{v}} t^{\mathbf{v}}$ by $s_{\mathbf{v}}$.

Lemma 3.14. *There is an isomorphism of rings* $\mathrm{PSS}_{log} : \mathbf{k}[s_1] \to SH^0(X)$.

Proof. Consider the spectral sequence of Corollary 3.3. An easy modification of this corollary shows that the ring structure on symplectic cohomology respects the filtration and hence this is a spectral sequence of rings. The argument of Theorem 2.12 shows that on the first page of the spectral sequence, we have that $\mathrm{PSS}^{(\mathbf{v}_1+\mathbf{v}_2)}_{log}(s_{\mathbf{v}_1+\mathbf{v}_2}) = \mathrm{PSS}^{(\mathbf{v}_1)}_{log}(s_{\mathbf{v}_1}) \cdot \mathrm{PSS}^{(\mathbf{v}_2)}_{log}(s_{\mathbf{v}_2})$. We have $\mathrm{PSS}_{log}(s_1)^{\mathbf{v}} = \mathrm{PSS}_{log}(s_{\mathbf{v}})$ + lower order terms. It follows that $\mathrm{PSS}_{log}(s_1)$ generates the ring freely. ☐

Unlike in the topological setting, the PSS_{log} map is not compatible with the topological product defined in Definition 2.11. For example, it is not difficult to see by a modification of Theorem 2.12 that in the case where $M = \mathbb{P}^2$ and E is a smooth elliptic curve, that we have that $\mathrm{PSS}_{log}(s_1)^3 = \mathrm{PSS}_{log}(s_3) + 6$ (the number six arises here as the degree of the dual elliptic curve from classical algebraic geometry).

Remark 3.15. *While this note was being written,*[**S**], *which considers a related construction of symplectic cohomology classes in the setting of anticanonical pencils, appeared. In that setting, the complement X is not exact, however the normal bundle to D is trivial (giving rise to the necessary convexity at infinity) and a version of symplectic cohomology can be defined over a suitable Novikov ring. It is not difficult to modify our arguments to obtain a suitable version of Theorems 3.13 and Lemma 3.14 in that setting.*

References

[AS] Mohammed Abouzaid and Paul Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. **14** (2010), no. 2, 627–718. MR2602848 ↑6

[CFH] K. Cieliebak, A. Floer, and H. Hofer, *Symplectic homology. II. A general construction*, Math. Z. **218** (1995), no. 1, 103–122. MR1312580 (95m:58055) ↑1

[CM] Kai Cieliebak and Klaus Mohnke, *Symplectic hypersurfaces and transversality in Gromov-Witten theory*, J. Symplectic Geom. **5** (2007), no. 3, 281–356. MR2399678 (2009j:53120) ↑8

[D] Luis Diogo, *Filtered Floer and Symplectic homology via Gromov-Witten theory*, Ph.D. Thesis, 2012. ↑2

[DL] Luis Diogo and Sam Lisi, *in preparation.* ↑2

[FH] A. Floer and H. Hofer, *Symplectic homology. I. Open sets in* $\mathbf{C}^n$, Math. Z. **215** (1994), no. 1, 37–88. MR1254813 (95b:58059) ↑1

[GP1] Sheel Ganatra and Daniel Pomerleano, *The Log PSS map is an isomorphism.* In preparation. ↑2

[GP2] ______, *A Log PSS map with applications to lagrangian embeddings*, 2016. Preprint. ↑1, 2, 7, 8, 9, 10

[IP] Eleny-Nicoleta Ionel and Thomas H. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. (2) **157** (2003), no. 1, 45–96. MR1954264 (2004a:53112) ↑11

[K] P. Kahn, *Pseudohomology and homology*, 2001. ↑7

[M] Mark McLean, *The growth rate of symplectic homology and affine varieties*, Geom. Funct. Anal. **22** (2012), no. 2, 369–442. MR2929069 ↑5

[MS] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2004. MR2045629 (2004m:53154) ↑12

[S] Paul Seidel, *Fukaya A_∞ structures associated to Lefschetz fibrations III*, 2016. ↑5, 9, 18

[V] C. Viterbo, *Functors and computations in Floer homology with applications. II*, 1996. Preprint. ↑1, 5